ASSOCIATED GRADED ALGEBRAS AND COALGEBRAS

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Abstract. We investigate the notion of associated graded coalgebra (algebra) of a bialgebra with respect to a subbialgebra (quotient bialgebra) and characterize those which are bialgebras of type one in the framework of abelian braided monoidal categories.

Contents

Introduction

Let H be a braided bialgebra in a cocomplete and complete abelian braided monoidal category $(\mathcal{M},c)$ satisfying $AB5$. Assume that the tensor product commutes with direct sums and is two-sided exact. Let M be in ${}_H^H\mathcal{M}_H^H$. Let $T=T_H(M)$ be the relative tensor algebra and let $T^c=T_H^c(M)$ be the relative cotensor coalgebra as introduced in [AMS1]. Then both T and T^c have a natural structure of graded braided bialgebra and the natural algebra morphism from T to T^c, which coincide with the canonical injections on H and M, is a graded bialgebra homomorphism. Thus its image is a graded braided bialgebra which is denoted by $H[M]$ and called *the braided bialgebra of type one associated to H and M*. Ordinary bialgebras of type one were introduced by Nichols in [Ni]. They came out to play a relevant role in the theory of Hopf Algebras. In particular, their "coinvariant" part, called Nichols algebra, has been deeply investigated, see e.g. [Ro], [AS] and the references therein.

Let $B\hookrightarrow E$ be a monomorphism in $\mathcal{M}$ which is a braided bialgebra homomorphism in $\mathcal{M}$. Under some technical assumptions, we prove in Theorem 4.7 that the following assertions are equivalent.

(1) gr_BE is the braided bialgebra of type one associated to B and $\frac{B\wedge_E B}{B}$.
(2) gr_BE is strongly $\mathbb{N}$-graded as an algebra (in the sense of Definition 3.2).
(3) $\oplus_{n\in\mathbb{N}}B^{\wedge_E n+1}$ is strongly $\mathbb{N}$-graded as an algebra.
(4) $B^{\wedge_E n+1}=(B\wedge_E B)^{\cdot_E n}$ for every $n\geq 2$.

Here gr_BE denotes the associated graded coalgebra $\oplus_{n\in\mathbb{N}}\frac{B^{\wedge_E n+1}}{B^{\wedge_E n}}$. As an application, in Corollary 4.8, we consider the case of a subbialgebra H of a bialgebra E over a field K.

Similar results are obtained in the case $gr_IE:=\oplus_{n\in\mathbb{N}}\frac{I^n}{I^{n+1}}$ where I is the kernel of an epimorphism $\pi:E\to B$ which is a braided bialgebra homomorphism in $\mathcal{M}$.

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The paper is organized as follows. In Section 1 we recall some definitions and introduce basic notations needed in the paper. In Section 2, we prove that the associated graded coalgebra $gr_C E := \oplus_{n\in\mathbb{N}} \frac{C^{\wedge_E n+1}}{C^{\wedge_E n}}$ for a given subcoalgebra C of a coalgebra E in $\mathcal{M}$, is a strongly $\mathbb{N}$-graded coalgebra (see Theorem 2.10) and hence it can be characterized as in Theorem 2.12. Dual results are obtained in Section 3 for the associated graded algebra $gr_I A := \oplus_{n\in\mathbb{N}} \frac{I^n}{I^{n+1}}$, where I is an ideal of an algebra A in $\mathcal{M}$. In Section 4, we firstly show that $gr_B E$ is a graded braided bialgebra in $(\mathcal{M}, c)$ (see Theorem 4.6) and then prove Theorem 4.7 which is the main result of the paper. Section 5 deals with the dual results. For the reader's sake, some technicalities are collected in Appendix A.

Finally we would like to outline that many results are firstly stated and proved in the coalgebra case where we found the proofs less straightforward. Also proofs in the algebra case are not given whenever they would have been an easy adaptation of the coalgebra ones.

In [AM2], results of the present paper are applied to study strictly graded bialgebras.

1. Preliminaries and Notations

Notations. Let $[(X, i_X)]$ be a subobject of an object E in an abelian category $\mathcal{M}$, where $i_X = i_X^E : X \hookrightarrow E$ is a monomorphism and $[(X, i_X)]$ is the associated equivalence class. By abuse of language, we will say that (X, i_X) is a subobject of E and we will write $(X, i_X) = (Y, i_Y)$ to mean that $(Y, i_Y) \in [(X, i_X)]$. The same convention applies to cokernels. If (X, i_X) is a subobject of E then we will write $(E/X, p_X) = \mathrm{Coker}(i_X)$, where $p_X = p_X^E : E \to E/X$.
Let $(X_1, i_{X_1}^{Y_1})$ be a subobject of Y_1 and let $(X_2, i_{X_2}^{Y_2})$ be a subobject of Y_2. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms such that $y \circ i_{X_1}^{Y_1} = i_{X_2}^{Y_2} \circ x$. Then there exists a unique morphism, which we denote by $y/x = \frac{y}{x} : Y_1/X_1 \to Y_2/X_2$, such that $\frac{y}{x} \circ p_{X_1}^{Y_1} = p_{X_2}^{Y_2} \circ y$:

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_{X_1}^{Y_1}} & Y_1 & \xrightarrow{p_{X_1}^{Y_1}} & \frac{Y_1}{X_1} \\ \downarrow x & & \downarrow y & & \downarrow \frac{y}{x} \\ X_2 & \xrightarrow{i_{X_2}^{Y_2}} & Y_2 & \xrightarrow{p_{X_2}^{Y_2}} & \frac{Y_2}{X_2} \end{array}$$

$\delta_{u,v}$ will denote the Kronecker symbol for every $u, v \in \mathbb{N}$.

1.1. **Monoidal Categories.** Recall that (see [Ka, Chap. XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$, $r_X : X \otimes \mathbf{1} \to X$, for every X, Y, Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom,* that is the following relation

$$(U \otimes a_{V,W,X}) \circ a_{U,V\otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W\otimes X} \circ a_{U\otimes V,W,X}$$

holds true, for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom,* that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in $\mathcal{M}$.

A *braided monoidal category* $(\mathcal{M}, c)$ is a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ equipped with a *braiding* c, that is a natural isomorphism $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ for every X, Y, Z in $\mathcal{M}$ satisfying

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) \qquad \text{and} \qquad c_{X,Y\otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z).$$

For further details on these topics, we refer to [Ka, Chapter XIII].

It is well known that the Pentagon Axiom completely solves the consistency problem arising out of the possibility of going from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$ in two different ways (see [Mj1, page 420]). This allows the notation $X_1 \otimes \cdots \otimes X_n$ forgetting the brackets for any object obtained from $X_1, \cdots X_n$ using $\otimes$. Also, as a consequence of the coherence theorem, the constraints take care of themselves and can then be omitted in any computation involving morphisms in $\mathcal{M}$.
Thus, for sake of simplicity, from now on, we will omit the associativity constraints.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be

introduced in the general setting of monoidal categories. Given an algebra A in $\mathcal{M}$ on can define the categories ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ of left, right and two-sided modules over A respectively. Similarly, given a coalgebra C in $\mathcal{M}$, one can define the categories of C-comodules ${}^C\mathcal{M}, \mathcal{M}^C, {}^C\mathcal{M}^C$. For more details, the reader is referred to [AMS2].

Definitions 1.2. Let $\mathcal{M}$ be a monoidal category.
We say that $\mathcal{M}$ is an **abelian monoidal category** if $\mathcal{M}$ is abelian and both the functors $X\otimes(-) : \mathcal{M} \to \mathcal{M}$ and $(-)\otimes X : \mathcal{M} \to \mathcal{M}$ are additive and right exact, for any $X \in \mathcal{M}$.
We say that $\mathcal{M}$ is an **coabelian monoidal category** if $\mathcal{M}^o$ is an abelian monoidal category, where $\mathcal{M}^o$ denotes the dual monoidal category of $\mathcal{M}$. Recall that $\mathcal{M}^o$ and $\mathcal{M}$ have the same objects but $\mathcal{M}^o(X,Y) = \mathcal{M}(Y,X)$ for any X, Y in $\mathcal{M}$.

Given an algebra A in an abelian monoidal category $\mathcal{M}$, there exist a suitable functor $\otimes_A$: ${}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A$ and constraints that make the category $({}_A\mathcal{M}_A, \otimes_A, A)$ abelian monoidal, see [AMS2, 1.11]. The tensor product over A in $\mathcal{M}$ of a right A-module V and a left A-module W is defined to be the coequalizer:

$$(V\otimes A)\otimes W \rightrightarrows V\otimes W \xrightarrow{{}_A\chi_{V,W}} V\otimes_A W \longrightarrow 0$$

Note that, since $\otimes$ preserves coequalizers, then $V\otimes_A W$ is also an A-bimodule, whenever V and W are A-bimodules.

Dually, let $\mathcal{M}$ be a coabelian monoidal category.
Given a coalgebra (C, Δ, ε) in $\mathcal{M}$, there exist of a suitable functor $\square_C : {}^C\mathcal{M}^C \times {}^C\mathcal{M}^C \to {}^C\mathcal{M}^C$ and constraints that make the category $({}^C\mathcal{M}^C, \square_C, C)$ coabelian monoidal. The cotensor product over C in $\mathcal{M}$ of a right C-bicomodule V and a left C-comodule W is defined to be the equalizer:

$$0 \longrightarrow V\square_C W \xrightarrow{{}_C\varsigma_{V,W}} V\otimes W \rightrightarrows V\otimes(C\otimes W)$$

Note that, since $\otimes$ preserves equalizers, then $V\square_C W$ is also a C-bicomodule, whenever V and W are C-bicomodules.

1.3. **Graded Objects.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of objects in a monoidal category $\mathcal{M}$ which is cocomplete abelian and let

$$X = \bigoplus_{n\in\mathbb{N}} X_n$$

be their coproduct in $\mathcal{M}$. In this case we also say that X is a *graded object of* $\mathcal{M}$ and that the sequence $(X_n)_{n\in\mathbb{N}}$ defines a graduation on X. A morphism

$$f : X = \bigoplus_{n\in\mathbb{N}} X_n \to Y = \bigoplus_{n\in\mathbb{N}} Y_n$$

is called a *graded homomorphism* whenever there exists a family of morphisms $(f_n : X_n \to Y_n)_{n\in\mathbb{N}}$ such that $f = \oplus_{n\in\mathbb{N}} f_n$ i.e. such that

$$f \circ i^X_{X_n} = i^Y_{Y_n} \circ f_n, \text{ for every } n \in \mathbb{N}.$$

We fix the following notations:

$$p_n : X \to X_n,$$

be the canonical projection and let

$$i_n : X_n \to X,$$

be the canonical injection for any $n \in \mathbb{N}$.

Given graded objects X, Y in $\mathcal{M}$ we set

$$(X\otimes Y)_n = \oplus_{a+b=n}(X_a \otimes Y_b).$$

Then this defines a graduation on $X\otimes Y$ whenever the tensor product commutes with direct sums. We denote by

$$X_a \otimes Y_b \overset{\gamma^{X,Y}_{a,b}}{\to} (X\otimes Y)_{a+b} \qquad \text{and} \qquad (X\otimes Y)_{a+b} \overset{\omega^{X,Y}_{a,b}}{\to} X_a \otimes Y_b$$

the canonical injection and projection respectively. We have

$$\sum_{a+b=n}\left(i_a^X\otimes i_b^Y\right)\omega_{a,b}^{X,Y} = \nabla\left[\left(i_a^X\otimes i_b^Y\right)_{a+b=n}\right] \tag{1}$$

$$\sum_{a+b=n}\gamma_{a,b}^{X,Y}\left(p_a^X\otimes p_b^Y\right) = \Delta\left[\left(p_a^X\otimes p_b^Y\right)_{a+b=n}\right] \tag{2}$$

where $\nabla\left[\left(i_a^X\otimes i_b^Y\right)_{a+b=n}\right]$ denotes the codiagonal morphism associated to the family $\left(i_a^X\otimes i_b^Y\right)_{a+b=n}$ and $\Delta\left[\left(p_a^X\otimes p_b^Y\right)_{a+b=n}\right]$ denotes the diagonal morphism associated to the family $\left(p_a^X\otimes p_b^Y\right)_{a+b=n}$.

2. The Associated Graded Coalgebra

2.1. Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums.

Recall that a *graded coalgebra* in $\mathcal{M}$ is a coalgebra (C,Δ,ε) where

$$C=\oplus_{n\in\mathbb{N}}C_n$$

is a graded object of $\mathcal{M}$ such that $\Delta:C\to C\otimes C$ is a graded homomorphism i.e. there exists a family $(\Delta_n)_{n\in\mathbb{N}}$ of morphisms

$$\Delta_n^C=\Delta_n:C_n\to(C\otimes C)_n=\oplus_{a+b=n}(C_a\otimes C_b)\text{ such that }\Delta=\oplus_{n\in\mathbb{N}}\Delta_n.$$

We set

$$\Delta_{a,b}^C=\Delta_{a,b}:=\left(C_{a+b}\overset{\Delta_{a+b}}{\to}(C\otimes C)_{a+b}\overset{\omega_{a,b}^{C,C}}{\to}C_a\otimes C_b\right).$$

A homomorphism $f:(C,\Delta_C,\varepsilon_C)\to(D,\Delta_D,\varepsilon_D)$ of coalgebras is a graded coalgebra homomorphism if it is a graded homomorphism too.

Definition 2.2. Let $(C=\oplus_{n\in\mathbb{N}}C_n,\Delta,\varepsilon)$ be a graded coalgebra in $\mathcal{M}$. In analogy with the group graded case (see [NT]), we say that C is a *strongly* $\mathbb{N}$*-graded coalgebra* whenever

$\Delta_{i,j}^C:C_{i+j}\to C_i\otimes C_j$ is a monomorphism for every $i,j\in\mathbb{N}$,

where $\Delta_{i,j}^C$ is the morphism defined in Definition 2.1.

Proposition 2.3. [AM1, Propositions 2.5 and 2.3] *Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums.*

1) Let $C=\oplus_{n\in\mathbb{N}}C_n$ be a graded object of $\mathcal{M}$ such that there exists a family $\left(\Delta_{a,b}^C\right)_{a,b\in\mathbb{N}}$

$$\Delta_{a,b}^C:C_{a+b}\to C_a\otimes C_b,$$

of morphisms and a morphism $\varepsilon_0^C:C_0\to\mathbf{1}$ which satisfy

$$\left(\Delta_{a,b}^C\otimes C_c\right)\circ\Delta_{a+b,c}^C=\left(C_a\otimes\Delta_{b,c}^C\right)\circ\Delta_{a,b+c}^C, \tag{3}$$

$$\left(C_d\otimes\varepsilon_0^C\right)\circ\Delta_{d,0}^C=r_{C_d}^{-1},\qquad\left(\varepsilon_0^C\otimes C_d\right)\circ\Delta_{0,d}^C=l_{C_d}^{-1}, \tag{4}$$

for every $a,b,c\in\mathbb{N}$. Then there exists a unique morphism $\Delta_C:C\to C\otimes C$ such that

$$(p_a^C\otimes p_b^C)\circ\Delta_C=\Delta_{a,b}^C\circ p_{a+b}^C,\ \text{for every } a,b\in\mathbb{N} \tag{5}$$

holds. Moreover $\left(C=\oplus_{n\in\mathbb{N}}C_n,\Delta_C,\varepsilon_C=\varepsilon_0^Cp_0^C\right)$ is a graded coalgebra.

2) If C is a graded coalgebra, then

$$\Delta_C\circ i_n^C=\sum_{a+b=n}\left(i_a^C\otimes i_b^C\right)\circ\Delta_{a,b}^C \tag{6}$$

holds, $\varepsilon_C=\varepsilon_Ci_0^Cp_0^C$ so that ε_C is a graded homomorphism, and we have that (3) and (4) hold for every $a,b,c\in\mathbb{N}$, where $\varepsilon_0^C=\varepsilon_Ci_0$.

Moreover $\left(C_0,\Delta_0=\Delta_{0,0}^C,\varepsilon_0^C=\varepsilon_Ci_0^C\right)$ is a coalgebra in $\mathcal{M}$, i_0^C is a coalgebra homomorphism and, for every $n\in\mathbb{N}$, $\left(C_n,\Delta_{0,n}^C,\Delta_{n,0}^C\right)$ is a C_0-bicomodule such that $p_n^C:C\to C_n$ is a morphism of C_0-bicomodules (C is a C_0-bicomodule through p_0^C).

Lemma 2.4. *Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums.*

Let $((C_a)_{a\in\mathbb{N}},(\beta_{C_a}^{C_b})_{a,b\in\mathbb{N}})$ be a direct system in $\mathcal{M}$, where, for $a\leq b$, $\beta_{C_a}^{C_b}:C_a\to C_b$ is an epimorphism. Assume that there exists a family $\left(\Delta_{a,b}^{C}\right)_{a,b\in\mathbb{N}}$

$$\Delta_{a,b}^{C}:C_{a+b}\to C_a\otimes C_b,$$

of morphisms and a morphism $\varepsilon_0^C:C_0\to\mathbf{1}$ which satisfy (3), (4),

$$\left(\beta_{C_a}^{C_{a+1}}\otimes C_b\right)\circ\Delta_{a,b}^{C}=\Delta_{a+1,b}^{C}\circ\beta_{C_{a+b}}^{C_{a+b+1}}\qquad\text{and}\qquad\left(C_a\otimes\beta_{C_b}^{C_{b+1}}\right)\circ\Delta_{a,b}^{C}=\Delta_{a,b+1}^{C}\circ\beta_{C_{a+b}}^{C_{a+b+1}}\tag{7}$$

for every $a,b,c\in\mathbb{N}$. Set $C_{-1}:=0$.

Let $\left(E_n,i_{E_n}^{C_n}\right):=\ker\left(\beta_{C_n}^{C_{n+1}}\right)$ for every $n\in\mathbb{N}$.

Then $C=\oplus_{n\in\mathbb{N}}C_n$ is a graded coalgebra, there is a unique coalgebra structure on $\oplus_{n\in\mathbb{N}}E_n$ such that

$$\oplus_{n\in\mathbb{N}}i_{E_n}^{C_n}:\oplus_{n\in\mathbb{N}}E_n\to\oplus_{n\in\mathbb{N}}C_n$$

is a coalgebra homomorphism and

1) *$E=\oplus_{n\in\mathbb{N}}E_n$ is a graded coalgebra such that $\oplus_{n\in\mathbb{N}}i_{E_n}^{C_n}$ is a graded homomorphism;*
2) *$\left(i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\right)\circ\Delta_{a,b}^{E}=\Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}$;*
3) *$\varepsilon_E=\varepsilon_0^C\circ i_{E_0}^{C_0}\circ p_0^E$.*

Proof. By Proposition 2.3, there exists a unique morphism $\Delta_C:C\to C\otimes C$ such that (6) holds. Moreover $\left(C=\oplus_{n\in\mathbb{N}}C_n,\Delta_C,\varepsilon_C=\varepsilon_0^Cp_0^C\right)$ is a graded coalgebra.

By left exactness of the tensor functors, we have the exact sequence

$$0\to E_a\otimes C_b\overset{i_{E_a}^{C_a}\otimes C_b}{\longrightarrow}C_a\otimes C_b\overset{\beta_{C_a}^{C_{a+1}}\otimes C_b}{\longrightarrow}C_{a+1}\otimes C_b.\tag{8}$$

From

$$\left(\beta_{C_a}^{C_{a+1}}\otimes C_b\right)\circ\Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}\overset{(7)}{=}\Delta_{a+1,b}^{C}\circ\beta_{C_{a+b}}^{C_{a+b+1}}\circ i_{E_{a+b}}^{C_{a+b}}=0.\tag{9}$$

and, by exactness of (8), there is a unique morphism $\alpha_{a,b}:E_{a+b}\to E_a\otimes C_b$ such that

$$\left(i_{E_a}^{C_a}\otimes C_b\right)\circ\alpha_{a,b}=\Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}.$$

By left exactness of the tensor functors, we have the exact sequence

$$0\to E_a\otimes E_b\overset{E_a\otimes i_{E_b}^{C_b}}{\longrightarrow}E_a\otimes C_b\overset{E_a\otimes\beta_{C_b}^{C_{b+1}}}{\longrightarrow}E_a\otimes C_{b+1}.\tag{10}$$

We obtain

$$\begin{aligned}\left(i_{E_a}^{C_a}\otimes C_{b+1}\right)\circ\left(E_a\otimes\beta_{C_b}^{C_{b+1}}\right)\circ\alpha_{a,b}&=\left(C_a\otimes\beta_{C_b}^{C_{b+1}}\right)\circ\left(i_{E_a}^{C_a}\otimes C_b\right)\circ\alpha_{a,b}\\&=\left(C_a\otimes\beta_{C_b}^{C_{b+1}}\right)\circ\Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}=0\end{aligned}$$

where the last equality is analogue to (9). Since $i_{E_a}^{C_a}\otimes C_{b+1}$ is a monomorphism, we deduce that $\left(E_a\otimes\beta_{C_b}^{C_{b+1}}\right)\circ\alpha_{a,b}=0$ so that, by exactness of (10), there is a unique morphism $\Delta_{a,b}^{E}:E_{a+b}\to E_a\otimes E_b$, such that

$$\left(E_a\otimes i_{E_b}^{C_b}\right)\circ\Delta_{a,b}^{E}=\alpha_{a,b}.$$

We compute

$$\left(i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\right)\circ\Delta_{a,b}^{E}=\left(i_{E_a}^{C_a}\otimes C_b\right)\circ\left(E_a\otimes i_{E_b}^{C_b}\right)\circ\Delta_{a,b}^{E}=\left(i_{E_a}^{C_a}\otimes C_b\right)\circ\alpha_{a,b}=\Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}$$

so that 2) holds true.

Let us prove that $\Delta_{a,b}^{E}$ fulfills (3). By 2), we infer

$$\left(i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\otimes i_{E_c}^{C_c}\right)\circ\left(\Delta_{a,b}^{E}\otimes E_c\right)\circ\Delta_{a+b,c}^{E}\quad=\quad\left(\Delta_{a,b}^{C}\otimes C_c\right)\circ\Delta_{a+b,c}^{C}\circ i_{E_{a+b+c}}^{C_{a+b+c}},$$

$$\left(i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\otimes i_{E_c}^{C_c}\right)\circ\left(E_a\otimes\Delta_{b,c}^{E}\right)\circ\Delta_{a,b+c}^{E} = \left(C_a\otimes\Delta_{b,c}^{C}\right)\circ\Delta_{a,b+c}^{C}\circ i_{E_{a+b+c}}^{C_{a+b+c}}.$$

Since C fulfills (3) and since $i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\otimes i_{E_c}^{C_c}$ is a monomorphism, we get (3) for E.

Let $\varepsilon_0^E := \varepsilon_0^C\circ i_{E_0}^{C_0}$. Let us prove that (4) hold for E. By 2), we have

$$\begin{aligned}\left(i_{E_d}^{C_d}\otimes\mathbf{1}\right)\circ\left(E_d\otimes\varepsilon_0^E\right)\circ\Delta_{d,0}^{E} &= \left(C_d\otimes\varepsilon_0^C\right)\circ\left(i_{E_d}^{C_d}\otimes i_{E_0}^{C_0}\right)\circ\Delta_{d,0}^{E} = \left(C_d\otimes\varepsilon_0^C\right)\circ\Delta_{d,0}^{C}\circ i_{E_d}^{C_d},\\ \left(i_{E_d}^{C_d}\otimes\mathbf{1}\right)\circ r_{E_d}^{-1} &= r_{C_d}^{-1}\circ i_{E_d}^{C_d}.\end{aligned}$$

By (4) for C and since $i_{E_d}^{C_d}\otimes\mathbf{1}$ is a monomorphism, we get the left equation of (4) for E. Similarly one gets the other equation. Thus, by applying Proposition 2.3, we conclude that E is a graded coalgebra and 3) holds true.

It remains to prove that $i := \oplus_{n\in\mathbb{N}} i_{E_n}^{C_n}$ is a coalgebra homomorphism. For every $a,b\in\mathbb{N}$, we have

$$\begin{aligned}&(i\otimes i)\circ\Delta_E\circ i_n^E \overset{(6)}{=} (i\otimes i)\circ\sum_{a+b=n}\left(i_a^E\otimes i_b^E\right)\circ\Delta_{a,b}^{E}\\ =&\sum_{a+b=n}\left(i_a^C\otimes i_b^C\right)\circ\left(i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\right)\circ\Delta_{a,b}^{E}\overset{2)}{=}\sum_{a+b=n}\left(i_a^C\otimes i_b^C\right)\circ\Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}\\ &\overset{(6)}{=}\Delta_C\circ i_n^C\circ i_{E_n}^{C_n} = \Delta_C\circ i\circ i_n^E\end{aligned}$$

and

$$\varepsilon_C\circ i\circ i_n^E = \varepsilon_C\circ i_n^C\circ i_{E_n}^{C_n} = \varepsilon_0^C\circ p_0^C\circ i_n^C\circ i_{E_n}^{C_n} = \delta_{n,0}\varepsilon_0^C\circ i_{E_0}^{C_0} = \varepsilon_0^C\circ i_{E_0}^{C_0}\circ p_0^E\circ i_n^E = \varepsilon_E\circ i_n^E$$

so that $(i\otimes i)\circ\Delta_E = \Delta_C\circ i$ and $\varepsilon_C\circ i = \varepsilon_E$. Thus i is a coalgebra homomorphism. □

LEMMA 2.5. *Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums.*

Let $((C_a)_{a\in\mathbb{N}},(\beta_{C_a}^{C_b})_{a,b\in\mathbb{N}})$ be an inverse system in $\mathcal{M}$, where, for $a\le b$, $\beta_{C_b}^{C_a}: C_b\to C_a$ is an epimorphism. Assume that there exists a family $\left(\Delta_{a,b}^{C}\right)_{a,b\in\mathbb{N}}$

$$\Delta_{a,b}^{C}: C_{a+b}\to C_a\otimes C_b,$$

of morphisms and a morphism $\varepsilon_0^C: C_0\to\mathbf{1}$ which satisfy (3), (4),

$$\left(\beta_{C_{a+1}}^{C_a}\otimes C_b\right)\circ\Delta_{a+1,b}^{C} = \Delta_{a,b}^{C}\circ\beta_{C_{a+b+1}}^{C_{a+b}}\quad\text{and}\quad\left(C_a\otimes\beta_{C_{b+1}}^{C_b}\right)\circ\Delta_{a,b+1}^{C} = \Delta_{a,b}^{C}\circ\beta_{C_{a+b+1}}^{C_{a+b}} \tag{11}$$

for every $a,b,c\in\mathbb{N}$. Set $C_{-1}:=0$.

Let $\left(E_n, i_{E_n}^{C_n}\right) := \ker\left(\beta_{C_n}^{C_{n-1}}\right)$ for every $n\in\mathbb{N}$.

Then $C=\oplus_{n\in\mathbb{N}}C_n$ is a graded coalgebra, there is a unique coalgebra structure on $\oplus_{n\in\mathbb{N}}E_n$ such that

$$\oplus_{n\in\mathbb{N}} i_{E_n}^{C_n}: \oplus_{n\in\mathbb{N}}E_n\to\oplus_{n\in\mathbb{N}}C_n$$

is a coalgebra homomorphism and

1) *$E=\oplus_{n\in\mathbb{N}}E_n$ is a graded coalgebra such that $\oplus_{n\in\mathbb{N}} i_{E_n}^{C_n}$ is a graded homomorphism;*
2) *$\left(i_{E_a}^{C_a}\otimes i_{E_b}^{C_b}\right)\circ\Delta_{a,b}^{E} = \Delta_{a,b}^{C}\circ i_{E_{a+b}}^{C_{a+b}}$;*
3) *$\varepsilon_E = \varepsilon_0^C\circ i_{E_0}^{C_0}\circ p_0^E$.*

Proof. It is similar to that of Lemma 2.4. □

Lemma 2.6. *Consider the following commutative diagram in an abelian category $\mathcal{C}$.*

$$\begin{tikzcd} 0 \arrow[r] & \ker(\beta) \arrow[r, "i"] & Z \arrow[r, "\beta"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ & & W \arrow[r, "\alpha"] & X \end{tikzcd}$$

Assume that both f and $g \circ i$ are monomorphisms. Then g is a monomorphism.

Proof. Let $\xi : T \to Z$ be a morphism such that $g \circ \xi = 0$. Then $f \circ \beta \circ \xi = \alpha \circ g \circ \xi = 0$ so that, since f is a monomorphism, we get $\beta \circ \xi = 0$. By the universal property of kernels ξ factors to a map $\overline{\xi} : T \to \ker(\beta)$ such that $i \circ \overline{\xi} = \xi$.

Now $g \circ i \circ \overline{\xi} = g \circ \xi = 0$ so that, since $g \circ i$ is a monomorphism, we get $\overline{\xi} = 0$. □

Theorem 2.7. *With hypothesis and notations of Lemma 2.5, the following assertions are equivalent.*

(1) $C = \oplus_{n \in \mathbb{N}} C_n$ *is a strongly* $\mathbb{N}$*-graded coalgebra.*
(2) $E = \oplus_{n \in \mathbb{N}} E_n$ *is a strongly* $\mathbb{N}$*-graded coalgebra.*

Proof. Let $((C_a)_{a \in \mathbb{N}}, (\beta^{C_b}_{C_a})_{a,b \in \mathbb{N}})$ be an inverse system in $\mathcal{M}$, where, for $a \leq b$, $\beta^{C_a}_{C_b} : C_b \to C_a$ is an epimorphism. Assume that there exists a family $\left(\Delta^C_{a,b}\right)_{a,b \in \mathbb{N}}$

$$\Delta^C_{a,b} : C_{a+b} \to C_a \otimes C_b,$$

of morphisms and a morphism $\varepsilon^C_0 : C_0 \to \mathbf{1}$ which satisfy (3), (4) and (11). Let $\left(E_n, i^{C_n}_{E_n}\right) := \ker\left(\beta^{C_{n-1}}_{C_n}\right)$ for every $n \in \mathbb{N}$.

(1) $\Rightarrow$ (2) It follows from $\left(i^{C_a}_{E_a} \otimes i^{C_b}_{E_b}\right) \circ \Delta^E_{a,b} = \Delta^C_{a,b} \circ i^{C_{a+b}}_{E_{a+b}}$ which holds in view of Lemma 2.5.

(2) $\Rightarrow$ (1) By assumption, $\Delta^E_{a,b}$ is a monomorphism for every $a, b \in \mathbb{N}$.

In view of [AM1, Theorem 2.22], it is enough to prove that $\Delta^C_{a,1} : C_{a+1} \to C_a \otimes C_1$ is a monomorphism by induction on $a \in \mathbb{N}$.

$a = 0$) From

$$(\varepsilon_0 \otimes C_1) \circ \Delta^C_{0,1} \overset{(4)}{=} l^{-1}_{C_1}$$

we deduce that $\Delta^C_{0,1}$ is a monomorphism.

$a - 1 \Rightarrow a$) Since $\Delta^E_{a,1}$ and $i^{C_a}_{E_a} \otimes i^{C_1}_{E_1}$ are monomorphisms and

$$\left(i^{C_a}_{E_a} \otimes i^{C_1}_{E_1}\right) \circ \Delta^E_{a,1} = \Delta^C_{a,1} \circ i^{C_{a+1}}_{E_{a+1}}$$

which holds in view of Lemma 2.5, we deduce that $\Delta^C_{a,1} \circ i^{C_{a+1}}_{E_{a+1}}$ is a monomorphism too. From (11), we get

$$\left(\beta^{C_{a-1}}_{C_a} \otimes C_1\right) \circ \Delta^C_{a,1} = \Delta^C_{a-1,1} \circ \beta^{C_a}_{C_{a+1}}.$$

Hence by inductive hypothesis, we can apply Lemma 2.6 to the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & E_{a+1} \arrow[r, "i^{C_{a+1}}_{E_{a+1}}"] & C_{a+1} \arrow[r, "\beta^{C_a}_{C_{a+1}}"] \arrow[d, "\Delta^C_{a,1}"'] & C_a \arrow[d, "\Delta^C_{a-1,1}"] \\ & & C_a \otimes C_1 \arrow[r, "\beta^{C_{a-1}}_{C_a} \otimes C_1"] & C_{a-1} \otimes C_1 \end{tikzcd}$$

□

2.8. Let $\mathcal{M}$ be a coabelian monoidal category.
Let (C, i^E_C) and (D, i^E_D) be two subobjects of a coalgebra (E, Δ, ε). Set

$$\Delta_{C,D} := (p^E_C \otimes p^E_D)\Delta : E \to \frac{E}{C} \otimes \frac{E}{D}$$

$$(C\wedge_E D, i^E_{C\wedge_E D}) = \ker(\Delta_{C,D}), \qquad i^E_{C\wedge_E D} : C\wedge_E D \to E$$

$$(\frac{E}{C\wedge_E D}, p^E_{C\wedge_E D}) = \mathrm{Coker}\left(i^E_{C\wedge_E D}\right) = \mathrm{Im}(\Delta_{C,D}), \qquad p^E_{C\wedge_E D} : E \to \frac{E}{C\wedge_E D}$$

Moreover, we have the following exact sequence:

$$0 \longrightarrow C\wedge_E D \overset{i^E_{C\wedge_E D}}{\longrightarrow} E \overset{p^E_{C\wedge_E D}}{\longrightarrow} \frac{E}{C\wedge_E D} \longrightarrow 0. \tag{12}$$

Since $(\frac{E}{C\wedge_E D}, p^E_{C\wedge_E D}) = \mathrm{Coker}\left(i^E_{C\wedge_E D}\right)$ and $\Delta_{C,D}\circ i^E_{C\wedge_E D} = 0$, by the universal property of the cokernel, there is a unique morphism $\overline{\Delta}_{C,D} : \frac{E}{C\wedge_E D} \to \frac{E}{C}\otimes\frac{E}{D}$ such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & C\wedge_E D \arrow[r, "i^E_{C\wedge_E D}"] & E \arrow[r, "p^E_{C\wedge_E D}"] \arrow[dr, "\Delta_{C,D}"'] & \frac{E}{C\wedge_E D} \arrow[r] \arrow[d, dotted, "\overline{\Delta}_{C,D}"] & 0 \\
 & & & \frac{E}{C}\otimes\frac{E}{D} &
\end{tikzcd}$$

is commutative. Since $(\frac{E}{C\wedge_E D}, p^E_{C\wedge_E D}) = \mathrm{Im}(\Delta_{C,D})$, it comes out that $\overline{\Delta}_{C,D}$ is a monomorphism. Assume now that (C, i^E_C) and (D, i^E_D) are two subcoalgebras of (E,Δ,ε). Since $\Delta_{C,D}\in {}^E\mathcal{M}^E$, it is straightforward to prove that $C\wedge_E D$ is a coalgebra and that $i^E_{C\wedge_E D}$ is a coalgebra homomorphism. Consider the case $C = 0$.
Since p^E_D is a morphism in ${}^E\mathcal{M}$, we have

$$\Delta_{0,D} = (\mathrm{Id}_E\otimes p^E_D)\circ\Delta = \rho^l_{E/D}\circ p^E_D.$$

Since $\rho^l_{E/D}$ is a monomorphism, we deduce that

$$(0\wedge_E D, i^E_{0\wedge_E D}) = \ker(\Delta_{0,D}) = \ker\left(p^E_D\right) = \left(D, i^E_D\right).$$

Analogously, in the case $D = 0$, one has

$$(C\wedge_E 0, i^E_{C\wedge_E 0}) = \left(C, i^E_C\right).$$

2.9. Let (C, i^E_C) be a subobject of a coalgebra (E,Δ,ε) in a coabelian monoidal category $\mathcal{M}$. We can define (see [AMS2]) the n-th wedge product $\left(C^{\wedge_E n}, i^E_{C^{\wedge_E n}}\right)$ of C in E where $i^E_{C^{\wedge_E n}} : C^{\wedge_E n}\to E$. By definition, we have

$$C^{\wedge_E 0} = 0 \qquad \text{and} \qquad C^{\wedge_E n} = C^{\wedge_E n-1}\wedge_E C, \text{ for every } n\geq 1.$$

One can check that $\left((C\wedge_E D)\wedge_E F, i^E_{(C\wedge_E D)\wedge_E F}\right)$ and $\left(C\wedge_E(D\wedge_E F), i^E_{C\wedge_E(D\wedge_E F)}\right)$ are isomorphic, for every subobject C, D, F of E, and thus can be identified. Therefore $C^{\wedge_E i}\wedge_E C^{\wedge_E j} = C^{\wedge_E i+j}$ and we can consider

$$\overline{\Delta}_{C^{\wedge_E i}, C^{\wedge_E j}} : \frac{E}{C^{\wedge_E i+j}} \to \frac{E}{C^{\wedge_E i}}\otimes\frac{E}{C^{\wedge_E j}}.$$

Assume now that (C, i^E_C) is a subcoalgebra of the coalgebra (E,Δ,ε). Then there is a (unique) coalgebra homomorphism

$$i^{C^{\wedge_E n+1}}_{C^{\wedge_E n}} : C^{\wedge_E n}\to C^{\wedge_E n+1}, \text{ for every } n\in\mathbb{N}.$$

such that $i^E_{C^{\wedge_E n+1}}\circ i^{C^{\wedge_E n+1}}_{C^{\wedge_E n}} = i^E_{C^{\wedge_E n}}$. We set

$$E_C := \oplus_{n\in\mathbb{N}}\frac{E}{C^{\wedge_E n}}$$

and

$$\Delta^{E_C}_{i,j} := \overline{\Delta}_{C^{\wedge_E i}, C^{\wedge_E j}} : \frac{E}{C^{\wedge_E i+j}} \to \frac{E}{C^{\wedge_E i}}\otimes\frac{E}{C^{\wedge_E j}}.$$

Since Δ is coassociative and by definition of $\Delta^{E_C}_{i,j}$, it is straightforward to prove that $\Delta^{E_C}_{i,j}$ fulfills

$$\left(\Delta^{E_C}_{a,b}\otimes\frac{E}{C^{\wedge_E c}}\right)\Delta^{E_C}_{a+b,c} = \left(\frac{E}{C^{\wedge_E a}}\otimes\Delta^{E_C}_{b,c}\right)\Delta^{E_C}_{a,b+c}. \tag{13}$$

$$\left(\frac{E}{C^{\wedge_E d}}\otimes\varepsilon\right)\Delta^{E_C}_{d,0}=r^{-1}_{\frac{E}{C^{\wedge_E d}}},\qquad\left(\varepsilon\otimes\frac{E}{C^{\wedge_E d}}\right)\Delta^{E_C}_{0,d}=l^{-1}_{\frac{E}{C^{\wedge_E d}}}\tag{14}$$

(15)

$$\left(\frac{E}{i^{C^{\wedge_E a+1}}_{C^{\wedge_E a}}}\otimes C_b\right)\circ\Delta^{E_C}_{a,b}=\Delta^{E_C}_{a+1,b}\circ\frac{E}{i^{C^{\wedge_E a+b+1}}_{C^{\wedge_E a+b}}},\qquad\left(C_a\otimes\frac{E}{i^{C^{\wedge_E b+1}}_{C^{\wedge_E b}}}\right)\circ\Delta^{E_C}_{a,b}=\Delta^{E_C}_{a,b+1}\circ\frac{E}{i^{C^{\wedge_E a+b+1}}_{C^{\wedge_E a+b}}}.$$

THEOREM 2.10. *Let $\mathcal{M}$ be a cocomplete coabelian monoidal category such that the tensor product commutes with direct sums.*

Let (C,i^E_C) be a subcoalgebra of a coalgebra (E,Δ,ε) in a coabelian monoidal category $\mathcal{M}$. For every $n\in\mathbb{N}$, we set

$$gr^n_C E=\frac{C^{\wedge_E n+1}}{C^{\wedge_E n}}.$$

Then $E_C=\oplus_{n\in\mathbb{N}}\frac{E}{C^{\wedge_E n}}$ is a graded coalgebra, there is a unique coalgebra structure on $gr_C E:=\oplus_{n\in\mathbb{N}}gr^n_C E$ such that

$$\oplus_{n\in\mathbb{N}}\frac{i^E_{C^{\wedge_E n+1}}}{C^{\wedge_E n}}:gr_C E\to E_C$$

is a coalgebra homomorphism and

(1) *$gr_C E$ is a graded coalgebra such that $\oplus_{n\in\mathbb{N}}\frac{i^E_{C^{\wedge_E n+1}}}{C^{\wedge_E n}}$ is a graded homomorphism;*
(2)

$$\left(\frac{i^E_{C^{\wedge_E a+1}}}{C^{\wedge_E a}}\otimes\frac{i^E_{C^{\wedge_E b+1}}}{C^{\wedge_E b}}\right)\circ\Delta^{gr_C E}_{a,b}=\Delta^{E_C}_{a,b}\circ\frac{i^E_{C^{\wedge_E a+b+1}}}{C^{\wedge_E a+b}};\tag{16}$$

(3) *$\varepsilon_{gr_C E}=\varepsilon_E\circ i^E_C\circ p^{gr_C E}_0$.*

Moreover $gr_C E$ is a strongly $\mathbb{N}$-graded coalgebra.

Proof. By (13), (14) and (15), we can apply Lemma 2.4 to the family $(\frac{E}{C^{\wedge_E n}})_{n\in\mathbb{N}}$. It remains to prove the last assertion.

From 2), since both $\frac{i^E_{C^{\wedge_E a+b+1}}}{C^{\wedge_E a+b}}$ and $\Delta^{E_C}_{a,b}$ are monomorphisms, we get that $\Delta^{gr_C E}_{a,b}$ is a monomorphism too, for every $a,b\in\mathbb{N}$. Thus $gr_C E$ is a strongly $\mathbb{N}$-graded coalgebra. □

DEFINITION 2.11. Let (C,i^E_C) be a subcoalgebra of a coalgebra (E,Δ,ε) in a cocomplete coabelian monoidal category $\mathcal{M}$ such that the tensor product commutes with direct sums.

The strongly $\mathbb{N}$-graded coalgebra $gr_C E$ defined in Theorem 2.10 will be called the *associated graded coalgebra* (of E with respect to C).

THEOREM 2.12. *Let $\mathcal{M}$ be a cocomplete and complete coabelian monoidal category satisfying $AB5$ such that the tensor product commutes with direct sums. Let (C,i^E_C) be a subcoalgebra of a coalgebra (E,Δ,ε) in $\mathcal{M}$ and let $gr_C E$ be the associated graded coalgebra.*

Let

$$T^c:=T^c_C\left(\frac{C\wedge_E C}{C}\right)$$

be the cotensor coalgebra. Then there is a unique coalgebra homomorphism

$$\psi:gr_C E\to T^c_C\left(\frac{C\wedge_E C}{C}\right),$$

such that $p^{T^c}_0\circ\psi=p^{gr_C E}_0$ and $p^{T^c}_1\circ\psi=p^{gr_C E}_1$.
Moreover ψ is a graded coalgebra homomorphism with

$$\psi_m=\left(p^{gr_C E}_1\right)^{\square m}\circ\overline{\Delta}^{m-1}_{gr_C E}\circ i^{gr_C E}_m\text{ for every }m\in\mathbb{N}$$

and the following equivalent assertions hold.
(a) *$gr_C E$ is a strongly $\mathbb{N}$-graded coalgebra.*
(a′) *$\Delta^{gr_C E}_{a,1}:gr^{a+1}_C(E)\to gr^a_C(E)\otimes gr^1_C(E)$ is a monomorphism for every $a\in\mathbb{N}$.*
(b) *ψ_n is a monomorphism for every $n\in\mathbb{N}$.*

(c) ψ *is a monomorphism.*
(d) $\oplus_{0\leq i\leq n-1} gr^i_C E = C^{\wedge^n_{gr_C E}}$, *for every* $n \geq 1$.
(e) $\oplus_{0\leq i\leq 1} gr^i_C E = C \oplus \frac{C\wedge_E C}{C} = C^{\wedge^2_{gr_C E}}$.

Proof. By Theorem 2.10, (a) holds. We conclude by applying [AM1, Theorem 2.22]. □

3. The Associated Graded Algebra

3.1. Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums.

Recall that a *graded algebra* in $\mathcal{M}$ is an algebra (A, m, u) where

$$A = \oplus_{n\in\mathbb{N}} A_n$$

is a graded object of $\mathcal{M}$ such that $m : A \otimes A \to A$ is a graded homomorphism i.e. there exists a family $(m_n)_{n\in\mathbb{N}}$ of morphisms

$$m^A_n = m_n : \oplus_{a+b=n}(A_a \otimes A_b) = (A\otimes A)_n \to A_n \text{ such that } m = \oplus_{n\in\mathbb{N}} m_n.$$

We set

$$m^A_{a,b} := \left(A_a \otimes A_b \overset{\gamma^{A,A}_{a,b}}{\to} (A\otimes A)_{a+b} \overset{m_{a+b}}{\to} A_{a+b} \right).$$

A homomorphism $f : (A, m_A, u_A) \to (B, m_B, u_B)$ of algebras is a graded algebra homomorphism if it is a graded homomorphism too.

Definition 3.2. Let $(A = \oplus_{n\in\mathbb{N}} A_n, m, u)$ be a graded algebra in $\mathcal{M}$. In analogy with the group graded case, we say that A is a *strongly* $\mathbb{N}$*-graded algebra* whenever
$m^A_{i,j} : A_i \otimes A_j \to A_{i+j}$ is an epimorphism for every $i, j \in \mathbb{N}$,
where $m^A_{i,j}$ is the morphism of Definition 3.1.

Proposition 3.3. [AM1, Proposition 3.4] *Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums.*

1) Let $A = \oplus_{n\in\mathbb{N}} A_n$ be a graded object of $\mathcal{M}$ such that there exists a family $(m_{a,b})_{a,b\in\mathbb{N}}$

$$m^A_{a,b} : A_a \otimes A_b \to A_{a+b},$$

of morphisms and a morphism $u^A_0 : \mathbf{1} \to A_0$ which satisfy

$$m^A_{a+b,c} \circ \left(m^A_{a,b} \otimes A_c\right) = m^A_{a,b+c} \circ \left(A_a \otimes m^A_{b,c}\right), \tag{17}$$

$$m^A_{d,0} \circ \left(A_d \otimes u^A_0\right) = r_{A_d}, \qquad m^A_{0,d} \circ \left(u^A_0 \otimes A_d\right) = l_{A_d}, \tag{18}$$

for every $a, b, c \in \mathbb{N}$. Then there exists a unique morphism $m_A : A \otimes A \to A$ such that

$$m_A \circ (i^A_a \otimes i^A_b) = i^A_{a+b} \circ m^A_{a,b}, \text{ for every } a, b \in \mathbb{N} \tag{19}$$

holds.
Moreover $\left(A = \oplus_{n\in\mathbb{N}} A_n, m_A, u_A = i^A_0 \circ u^A_0\right)$ is a graded algebra.

2) If A is a graded algebra then

$$p^A_{a+b} \circ m_A = \sum_{a+b=n} m^A_{a,b} \circ \left(p^A_a \otimes p^A_b\right) \tag{20}$$

holds, $u_A = i^A_0 p^A_0 u_A$ so that u_A is a graded homomorphism, and we have that (17) and (18) hold for every $a, b, c \in \mathbb{N}$, where $u^A_0 = p^A_0 u_A$.

Moreover $\left(A_0, m_0 = m^A_{0,0}, u^A_0 = p^A_0 u_A\right)$ is an algebra in $\mathcal{M}$, p^A_0 is an algebra homomorphism and, for every $n \in \mathbb{N}$, $\left(A_n, m^A_{0,n}, m^A_{n,0}\right)$ is an A_0-bimodule such that $i^A_n : A_n \to A$ is a morphism of A_0-bimodules (A is an A_0-bimodule through i^A_0).

LEMMA 3.4. *Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums. Let $((A_a)_{a\in\mathbb{N}},(a_{A_a}^{A_b})_{a,b\in\mathbb{N}})$ be a direct system in $\mathcal{M}$ where, for $a\leq b$, $i_{A_a}^{A_b}:A_a\to A_b$ is a monomorphism. Assume that there exists a family $\left(m_{a,b}^A\right)_{a,b\in\mathbb{N}}$*

$$m_{a,b}^A:A_a\otimes A_b\to A_{a+b},$$

of morphisms and a morphism $u_0^A:\mathbf{1}\to A_0$ which satisfy (17), (18),

$$(21)\quad m_{a+1,b}^A\circ\left(i_{A_a}^{A_{a+1}}\otimes A_b\right)=i_{A_{a+b}}^{A_{a+b+1}}\circ m_{a,b}^A\qquad\text{and}\qquad m_{a,b+1}^A\circ\left(A_a\otimes i_{A_b}^{A_{b+1}}\right)=i_{A_{a+b}}^{A_{a+b+1}}\circ m_{a,b}^A$$

for every $a,b,c\in\mathbb{N}$. Set $A_{-1}:=0$.

Then $A=\oplus_{n\in\mathbb{N}}A_n$ is a graded algebra and there are unique algebra structure on $\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n-1}}$ such that

$$\oplus_{n\in\mathbb{N}}p_{A_{n-1}}^{A_n}:\oplus_{n\in\mathbb{N}}A_n\to\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n-1}}$$

is an algebra homomorphism. Moreover

1) *$E=\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n-1}}$ is a graded algebra such that $\oplus_{n\in\mathbb{N}}p_{A_{n-1}}^{A_n}$ is a graded homomorphism;*
2) *$m_{a,b}^E\circ\left(p_{A_{a-1}}^{A_a}\otimes p_{A_{b-1}}^{A_b}\right)=p_{A_{a+b-1}}^{A_{a+b}}\circ m_{a,b}^A$;*
3) *$u_E=i_0^E\circ p_{A_{-1}}^{A_0}\circ u_0^A$.*

Proof. It is analogous to that of Lemma 2.4. □

LEMMA 3.5. *Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums. Let $((A_a)_{a\in\mathbb{N}},(i_{A_a}^{A_b})_{a,b\in\mathbb{N}})$ be an inverse system in $\mathcal{M}$ where, for $a\leq b$, $i_{A_b}^{A_a}:A_b\to A_a$ is a monomorphism. Assume that there exists a family $\left(m_{a,b}^A\right)_{a,b\in\mathbb{N}}$*

$$m_{a,b}^A:A_a\otimes A_b\to A_{a+b},$$

of morphisms and a morphism $u_0^A:\mathbf{1}\to A_0$ which satisfy (17), (18),

$$(22)\quad m_{a,b}^A\circ\left(i_{A_{a+1}}^{A_a}\otimes A_b\right)=i_{A_{a+b+1}}^{A_{a+b}}\circ m_{a+1,b}^A\qquad\text{and}\qquad m_{a,b}^A\circ\left(A_a\otimes i_{A_{b+1}}^{A_b}\right)=i_{A_{a+b+1}}^{A_{a+b}}\circ m_{a,b+1}^A$$

for every $a,b,c\in\mathbb{N}$. Set $A_{-1}:=0$.

Then $A=\oplus_{n\in\mathbb{N}}A_n$ is a graded algebra and there are unique algebra structure on $\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n+1}}$ such that

$$\oplus_{n\in\mathbb{N}}p_{A_{n+1}}^{A_n}:\oplus_{n\in\mathbb{N}}A_n\to\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n+1}}$$

is an algebra homomorphism. Moreover

1) *$E=\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n+1}}$ is a graded algebra such that $\oplus_{n\in\mathbb{N}}p_{A_{n+1}}^{A_n}$ is a graded homomorphism;*
2) *$m_{a,b}^E\circ\left(p_{A_{a+1}}^{A_a}\otimes p_{A_{b+1}}^{A_b}\right)=p_{A_{a+b+1}}^{A_{a+b}}\circ m_{a,b}^A$;*
3) *$u_E=i_0^E\circ p_{A_1}^{A_0}\circ u_0^A$.*

Proof. It is similar to that of Lemma 3.4. □

THEOREM 3.6. *With hypothesis and notations of Lemma 3.4, the following assertions are equivalent.*

(1) *$A=\oplus_{n\in\mathbb{N}}A_n$ is a strongly $\mathbb{N}$-graded algebra.*
(2) *$E=\oplus_{n\in\mathbb{N}}\frac{A_n}{A_{n-1}}$ is a strongly $\mathbb{N}$-graded algebra.*

Proof. It is analogous to that of Theorem 2.7. □

3.7. Recall from [AMS2] that an *ideal* of an algebra (A,m,u) in a monoidal category $(\mathcal{M},\otimes,\mathbf{1})$ is a pair (I,i_I^A) where I is an A-bimodule and

$$i_I^A:I\to A$$

is a morphism of A-bimodules which is a monomorphism in $\mathcal{M}$.
A morphism $f : I \to J$ in ${}_A\mathcal{M}_A$, where I,J are two ideals, is called a *morphism of ideals* whenever

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr, "i_I^A"'] & & J \arrow[dl, "i_J^A"] \\ & A & \end{tikzcd}$$

Note that f is a monomorphism in $\mathcal{M}$ as i_I^A is a monomorphism. Moreover f is unique, as i_J^A is a monomorphism.

3.8. Let $\mathcal{M}$ be an abelian monoidal category.
Let (I, i_I^A) and (J, i_J^A) be two subobjects of an algebra (A, m, u). Set

$$m_{I,J} := m(i_I^A \otimes i_J^A) : I \otimes J \to A$$
$$(Q_{I,J}, \pi_{I,J}) = \mathrm{Coker}\,(m_{I,J})\,, \qquad \pi_{I,J}^A : A \to Q_{I,J}$$
$$(IJ, i_{IJ}^A) = \mathrm{Ker}\left(\pi_{I,J}^A\right) = \mathrm{Im}\,(m_{I,J})\,, \qquad i_{IJ}^A : IJ \to A$$

The subobject (IJ, i_{IJ}^A) of A is called *the product of I and J*.
Moreover, we have the following exact sequence:

$$0 \longrightarrow IJ \xrightarrow{i_{IJ}^A} A \xrightarrow{\pi_{I,J}^A} Q_{I,J} \longrightarrow 0. \tag{23}$$

Since $(IJ, i_{IJ}^A) = \mathrm{Ker}\left(\pi_{I,J}^A\right)$ and $\pi_{I,J}^A m_{I,J} = 0$, by the universal property of the kernel, there is a unique morphism $\overline{m}_{I,J} : I \otimes J \to IJ$ such that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & IJ \arrow[rr, "i_{IJ}^A"] & & A \arrow[r, "\pi_{I,J}^A"] & Q_{I,J} \arrow[r] & 0 \\ & & I \otimes J \arrow[ul, dotted, "\overline{m}_{I,J}"] \arrow[ur, "m_{I,J}"'] & & & \end{tikzcd}$$

is commutative. Since IJ =Im$(m_{I,J})$, it comes out that $\overline{m}_{I,J}$ is an epimorphism.
Consider the case $I = A$.

Assume now that (I, i_I^A) and (J, i_J^A) are two ideals of (A, m, u).

In this case (IJ, i_{IJ}^A) is an ideal of A and $m_{I,J} \in {}_A\mathcal{M}_A$ so that it is straightforward to prove that $Q_{I,J}$ is an algebra and that $\pi_{I,J}$ and algebra homomorphism.
Since i_J^A is a morphism in ${}_A\mathcal{M}$, we have

$$m_{A,J} = m \circ (\mathrm{Id}_A \otimes i_J^A) = i_J^A \circ \mu_J^l.$$

Since i_J^A is a monomorphism and μ_J^l an epimorphism, we deduce that

$$(AJ, iv_{AJ}) = \mathrm{Im}\,(m_{A,J}) = \left(J, i_J^A\right).$$

Analogously, in the case $J = A$, one has

$$(IA, i_{IA}^A) = \left(I, i_I^A\right).$$

3.9. Let A be an algebra in an abelian monoidal category $\mathcal{M}$ and let (I, i_I^A) be a subobject of A. We can define (see [AMS2]) the n-th power $\left(I^n, i_{I^n}^A\right)$ of I, where $i_{I^n}^A : I^n \to A$. By Definition

$$I^0 = A \qquad \text{and} \qquad I^n = I^{n-1}I, \text{ for every } n \geq 1.$$

For every subobject I, J, K of A one can check that $\left((IJ)\,K, i_{(IJ)K}^A\right)$ and $\left(I\,(JK)\,, i_{I(JK)}^A\right)$ are isomorphic and thus can be identified. Therefore $I^iI^j = I^{i+j}$ and we can consider $\overline{m}_{I^i,I^j} : I^i \otimes I^j \to I^{i+j}$. We set

$$m_{i,j}^I := \overline{m}_{I^i,I^j} : I^i \otimes I^j \to I^{i+j}.$$

Assume now that (I, i_I^A) is an ideal of A. Then there is a (unique) morphism of ideals

$$i_{I^{n+1}}^{I^n} : I^{n+1} \to I^n, \text{ for every } n \in \mathbb{N}.$$

Since m is associative and by definition of $m_{i,j}^I$, it is straightforward to prove that $m_{i,j}^I$ fulfills

$$m_{a+b,c}^I \left(m_{a,b}^I \otimes I^c \right) = m_{a,b+c}^I \left(I^a \otimes m_{b,c}^I \right). \tag{24}$$

$$m_{d,0}^I \left(I^d \otimes u \right) = r_{I^d}, \qquad m_{0,d}^I \left(u \otimes I^d \right) = l_{I^d} \tag{25}$$

$$m_{a,b}^I \circ \left(i_{I^{a+1}}^{I^a} \otimes I^b \right) = i_{I^{a+b+1}}^{I^{a+b}} \circ m_{a+1,b}^I, \qquad m_{a,b}^I \circ \left(I^a \otimes i_{I^{b+1}}^{I^b} \right) = i_{I^{a+b+1}}^{I^{a+b}} \circ m_{a,b+1}^I \tag{26}$$

Theorem 3.10. *Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums.*

Let (A, m, u) be an algebra in $\mathcal{M}$ and let (I, i_I) be an ideal of A in $\mathcal{M}$. For every $n \in \mathbb{N}$, we set

$$gr_I^n A = \frac{I^n}{I^{n+1}}.$$

Then $\oplus_{n\in\mathbb{N}} I^n$ is a graded algebra and there are unique algebra structure on $gr_I A := \oplus_{n\in\mathbb{N}} gr_I^n A$ such that

$$\oplus_{n\in\mathbb{N}} p_{I^{n+1}}^{I^n} : \oplus_{n\in\mathbb{N}} I^n \to gr_I A$$

is an algebra homomorphism. Moreover

1) *$gr_I A$ is a graded algebra such that $\oplus_{n\in\mathbb{N}} \pi_n^I$ is a graded homomorphism;*
2)

$$m_{a,b}^{gr_I A} \circ \left(p_{I^{a+1}}^{I^a} \otimes p_{I^{b+1}}^{I^b} \right) = p_{I^{a+b+1}}^{I^{a+b}} \circ m_{a,b}^I, \text{ for every } a, b \in \mathbb{N} \tag{27}$$

3) *$u_{gr_I A} = i_0^{gr_I A} \circ p_I^A \circ u$*

Moreover $gr_I A$ is a strongly $\mathbb{N}$-graded algebra.

Proof. By (24), (25) and (26), we can apply Lemma 3.5 to the family $(I^n)_{n\in\mathbb{N}}$. It remains to prove the last assertion. From 2), since both $p_{I^{a+b+1}}^{I^{a+b}}$ and $m_{a,b}^I$ are epimorphisms, we get that $m_{a,b}^{gr_I A}$ is an epimorphism too, for every $a, b \in \mathbb{N}$. Thus $gr_I A$ is a strongly $\mathbb{N}$-graded algebra. □

Definition 3.11. Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums.

Given an ideal I of an algebra A in $\mathcal{M}$, the strongly $\mathbb{N}$-graded algebra $gr_I A$ defined in Theorem 3.10 will be called the *associated graded algebra* (of A with respect to I).

Theorem 3.12. *Let $\mathcal{M}$ be a cocomplete abelian monoidal category such that the tensor product commutes with direct sums. Let I be an ideal of an algebra A in $\mathcal{M}$ an let $gr_I A$ be the associated graded algebra.*

Let

$$T := T_{\frac{A}{I}}\left(\frac{I}{I^2}\right)$$

be the tensor algebra. Then there is a unique algebra homomorphism

$$\varphi : T_{\frac{A}{I}}\left(\frac{I}{I^2}\right) \to gr_I A,$$

such that $\varphi \circ i_0^T = i_0^{gr_I A}$ and $\varphi \circ i_1^T = i_1^{gr_I A}$.
Moreover φ is a graded algebra homomorphism with

$$\varphi_t = p_t^{gr_I A} \circ \overline{m}_{gr_I A}^{t-1} \circ \left(i_1^{gr_I A} \right)^{\otimes_{A_0} t} \text{ for every } t \in \mathbb{N}$$

and the following equivalent assertions hold.

(a) *$gr_I A$ is a strongly $\mathbb{N}$-graded algebra.*
(a′) *$m_{a,1}^{gr_I A} : gr_I^a A \otimes gr_I^1 A \to gr_I^{a+1} A$ is an epimorphism for every $a \in \mathbb{N}$.*
(b) *φ_n is an epimorphism for every $n \in \mathbb{N}$.*
(c) *φ is an epimorphism.*
(d) *$\oplus_{i\geq n} gr_I^i A = \left(\oplus_{i\geq 1} gr_I^i A \right)^n$, for every $n \in \mathbb{N}$.*

(e) $\oplus_{i\geq 2} gr_I^i A = \left(\oplus_{i\geq 1} gr_I^i A\right)^2$.

Proof. By Theorem 3.10, (a) holds. We conclude by applying [AM1, Theorem 3.11]. □

4. The Associated Graded Coalgebra of a Bialgebra With Respect to a Subbialgebra

Lemma 4.1. *Let $(\mathcal{M},c)$ be a cocomplete abelian braided monoidal category such that the tensor product commutes with direct sums. Consider in $(\mathcal{M},c)$ a datum $(B,m_B,u_B,\Delta_B,\varepsilon_B)$ consisting of a graded object B with graduation defined by $(B_k)_{k\in\mathbb{N}}$ such that, with respect to this graduation,*

- *(B,m_B,u_B) is a graded algebra and*
- *$(B,\Delta_B,\varepsilon_B)$ is a graded coalgebra.*

Assume that for every $a,b\in\mathbb{N}$,

$$\begin{aligned}&\sum_{s+t=a+b}(i_s\otimes i_t)\,\Delta_{s,t}m_{a,b}\\ = &\sum_{s'+t'=a}\sum_{s''+t''=b}(i_{s'+s''}\otimes i_{t'+t''})\circ(m_{s',s''}\otimes m_{t',t''})\circ\left(B_{s'}\otimes c_{B_{t'},B_{s''}}\otimes B_{t''}\right)\circ(\Delta_{s',t'}\otimes\Delta_{s'',t''})\end{aligned}$$

$$\varepsilon_0 m_{0,0} = m_{\mathbf{1}}\left(\varepsilon_0\otimes\varepsilon_0\right).$$

Then B is a graded braided bialgebra in $(\mathcal{M},c)$.

Proof. It follows easily by using (19), (6). □

Theorem 4.2. *Let $\mathcal{M}$ be an coabelian monoidal category. Let $((X_i)_{i\in\mathbb{N}},(\xi_i^j)_{i,j\in\mathbb{N}})$ be a direct system in $\mathcal{M}$ where, for $i\leq j$, $\xi_i^j: X_i\to X_j$.*

Let $(\xi_i: X_i\to X)_{i\in\mathbb{N}}$ be a compatible family of morphisms with respect to the given direct system. Assume that

- *$\xi_i^j: X_i\to X_j$ is a split monomorphism for every $i\leq j$,*
- *$X_0=0$,*
- *$\xi_i: X_i\to X$ is a monomorphism for every $i\in\mathbb{N}$*

and denote by $\tau_i: X\to\frac{X}{X_i}$ the canonical projection for every $i\in\mathbb{N}$.

Then, for every $n\in\mathbb{N}$, the following sequence is exact.

$$\bigoplus_{a+b=n+1}X_a\otimes X_b\overset{\nabla\left[(\xi_a\otimes\xi_b)_{a+b=n+1}\right]}{\longrightarrow}X\otimes X\overset{\Delta\left[(\tau_a\otimes\tau_b)_{a+b=n}\right]}{\longrightarrow}\bigoplus_{a+b=n}\frac{X}{X_a}\otimes\frac{X}{X_b}.$$

Proof. Apply [AM3, Theorem 3.1]. □

Notations 4.3. *In this section, from now on, the following assumptions and notations will be used.*

$(\mathcal{M},c)$ is a cocomplete abelian coabelian braided monoidal category such that the tensor product commutes with direct sums.

Let $i_B^E: B\hookrightarrow E$ be a monomorphism in $\mathcal{M}$ which is a braided bialgebra homomorphism in $\mathcal{M}$ and let $i_{B^{\wedge_E n}}^E: B^{\wedge_E n}\to E$ and $i_{B^{\wedge_E a}}^{B^{\wedge_E b}}: B^{\wedge_E a}\to B^{\wedge_E b}$ $(a\leq b)$ be the canonical injections.

Assume that $i_{B^{\wedge_E a}}^{B^{\wedge_E b}}$ is a split monomorphism in $\mathcal{M}$ for every $a\leq b$.

By Theorem 4.2, we have the following exact sequence

$$\bigoplus_{a+b=n+1}B^{\wedge_E a}\otimes B^{\wedge_E b}\overset{\nabla\left[\left(i_{B^{\wedge_E a}}^E\otimes i_{B^{\wedge_E b}}^E\right)_{a+b=n+1}\right]}{\longrightarrow}E\otimes E\overset{\Delta\left[\left(p_{B^{\wedge_E a}}^E\otimes p_{B^{\wedge_E b}}^E\right)_{a+b=n}\right]}{\longrightarrow}\bigoplus_{a+b=n}\frac{E}{B^{\wedge_E a}}\otimes\frac{E}{B^{\wedge_E b}}.$$

Let

$$\left(\sum_{a+b=n+1}B^{\wedge_E a}\otimes B^{\wedge_E b},\beta_n\right)=\mathrm{Im}\left\{\nabla\left[\left(i_{B^{\wedge_E a}}^E\otimes i_{B^{\wedge_E b}}^E\right)_{a+b=n+1}\right]\right\}$$

and let

$$\gamma_n : \bigoplus_{a+b=n+1} B^{\wedge_E a} \otimes B^{\wedge_E b} \twoheadrightarrow \sum_{a+b=n+1} B^{\wedge_E a} \otimes B^{\wedge_E b}$$

be the unique morphism such that

$$\beta_n \circ \gamma_n = \nabla \left[\left(i^E_{B^{\wedge_E a}} \otimes i^E_{B^{\wedge_E b}} \right)_{a+b=n+1} \right]. \tag{28}$$

Since $\left(B^{\wedge_E a+b}, i^E_{B^{\wedge_E a+b}}\right) = \ker\left(\left(p^E_{B^{\wedge_E a}} \otimes p^E_{B^{\wedge_E b}}\right)\Delta_E\right)$, *we have* $\left(p^E_{B^{\wedge_E a}} \otimes p^E_{B^{\wedge_E b}}\right) \circ \Delta_E \circ i^E_{B^{\wedge_E a+b}} = 0$ *so that*

$$\Delta \left[\left(p^E_{B^{\wedge_E a}} \otimes p^E_{B^{\wedge_E b}} \right)_{a+b=n} \right] \circ \Delta_E \circ i^E_{B^{\wedge_E a+b}} = 0$$

and hence, by the exactness of the sequence above, there exists a unique morphism

$$\alpha_n : B^{\wedge_E n} \to \sum_{a+b=n+1} B^{\wedge_E a} \otimes B^{\wedge_E b}$$

such that

$$\Delta_E \circ i^E_{B^{\wedge_E n}} = \beta_n \circ \alpha_n, \text{ for every } n \in \mathbb{N}. \tag{29}$$

Lemma 4.4. *0) For every $s, t, u, v \in \mathbb{N}$, we have*

$$\begin{aligned} & \left(p^E_{B^{\wedge_E s}} \otimes p^E_{B^{\wedge_E t}}\right) \circ (m_E \otimes m_E) \circ (E \otimes c \otimes E) \circ (\beta_u \otimes \beta_v) \circ (\gamma_u \otimes \gamma_v) \\ = \ & \nabla \left[\left(\begin{array}{c} \left(\left[p^E_{B^{\wedge_E s}} m_E \left(i^E_{B^{\wedge_E a}} \otimes i^E_{B^{\wedge_E b}}\right)\right] \otimes \left[p^E_{B^{\wedge_E t}} m_E \left(i^E_{B^{\wedge_E c}} \otimes i^E_{B^{\wedge_E d}}\right)\right] \right) \\ \circ \left(B^{\wedge_E a} \otimes c_{B^{\wedge_E c}, B^{\wedge_E b}} \otimes B^{\wedge_E d}\right) \end{array} \right)_{\substack{a+c=u+1 \\ b+d=v+1}} \right] \end{aligned} \tag{30}$$

1) The following relations hold.

$$p^E_{B^{\wedge_E u+v-1}} m_E \left(i^E_{B^{\wedge_E u}} \otimes i^E_{B^{\wedge_E v}}\right) = 0 \text{ for every } u, v \in \mathbb{N}, u+v \geq 1. \tag{31}$$

2) For every $a, b \in \mathbb{N}$, there exists a unique morphism $m^{a,b}_{\wedge} : B^{\wedge_E a+1} \otimes B^{\wedge_E b+1} \to B^{\wedge_E a+b+1}$ such that

$$i^E_{B^{\wedge_E a+b+1}} \circ m^{a,b}_{\wedge} = m_E \left(i^E_{B^{\wedge_E a+1}} \otimes i^E_{B^{\wedge_E b+1}}\right), \text{ for every } a, b \in \mathbb{N}. \tag{32}$$

3) For every $a, b, c, d \in \mathbb{N}$, we have

$$m^{a+b,c}_{\wedge} \circ \left(m^{a,b}_{\wedge} \otimes B^{\wedge_E c+1}\right) = m^{a,b+c}_{\wedge} \circ \left(B^{\wedge_E a+1} \otimes m^{b,c}_{\wedge}\right),$$

$$m^{d,0}_{\wedge} \circ \left(B^{\wedge_E d+1} \otimes u_B\right) = r_{B^{\wedge_E d+1}}, \qquad m^{0,d}_{\wedge} \circ \left(u_B \otimes B^{\wedge_E d+1}\right) = l_{B^{\wedge_E d+1}},$$

$$m^{a+1,b}_{\wedge} \circ \left(i^{B^{\wedge_E a+2}}_{B^{\wedge_E a+1}} \otimes B^{\wedge_E b+1}\right) = i^{B^{\wedge_E a+b+2}}_{B^{\wedge_E a+b+1}} \circ m^{a,b}_{\wedge}, \qquad m^{a,b+1}_{\wedge} \circ \left(B^{\wedge_E a+1} \otimes i^{B^{\wedge_E b+2}}_{B^{\wedge_E b+1}}\right) = i^{B^{\wedge_E a+b+2}}_{B^{\wedge_E a+b+1}} \circ m^{a,b}_{\wedge}$$

Proof. See the Appendix. □

Proposition 4.5. *$\oplus_{n\in\mathbb{N}} B^{\wedge_E n+1}$ is a graded algebra and there are unique algebra structure on $gr_B E = \oplus_{n\in\mathbb{N}} gr^n_B E$ such that*

$$\oplus_{n\in\mathbb{N}} p^{B^{\wedge_E n+1}}_{B^{\wedge_E n}} : \oplus_{n\in\mathbb{N}} B^{\wedge_E n+1} \to gr_B E$$

is an algebra homomorphism. Moreover

1) *$gr_B E = \oplus_{n\in\mathbb{N}} gr^n_B E$ is a graded algebra such that $\oplus_{n\in\mathbb{N}} p^{B^{\wedge_E n+1}}_{B^{\wedge_E n}}$ is a graded homomorphism;*
2)

$$p^{B^{\wedge_E a+b+1}}_{B^{\wedge_E a+b}} \circ m^{a,b}_{\wedge} = m^{gr_B E}_{a,b} \circ \left(p^{B^{\wedge_E a+1}}_{B^{\wedge_E a}} \otimes p^{B^{\wedge_E b+1}}_{B^{\wedge_E b}}\right). \tag{33}$$

3) $u_{gr_B E} = i^{gr_B E}_0 \circ p^B_{B^{\wedge_E 0}} \circ u_B$

Proof. By Lemma 4.4, we can apply Lemma 3.4 to the family $(B^{\wedge_E n+1})_{n\in\mathbb{N}}$. □

THEOREM 4.6. *Let $(\mathcal{M}, c)$ be a cocomplete and complete abelian coabelian braided monoidal category satisfying $AB5$. Assume that the tensor product commutes with direct sums. Let $i_B^E : B \hookrightarrow E$ be a monomorphism in $\mathcal{M}$ which is a braided bialgebra homomorphism in $\mathcal{M}$ and let $i_{B^{\wedge_E n}}^E : B^{\wedge_E n} \to E$ and $i_{B^{\wedge_E a}}^{B^{\wedge_E b}} : B^{\wedge_E a} \to B^{\wedge_E b}$ $(a \leq b)$ be the canonical injections. Assume that $i_{B^{\wedge_E a}}^{B^{\wedge_E b}}$ is a split monomorphism in $\mathcal{M}$ for every $a \leq b$.*
Then $gr_B E$ is a graded braided bialgebra in $(\mathcal{M}, c)$.

Proof. Set $i_n := i_{B^{\wedge_E a}}^E$, $p_n := p_{B^{\wedge_E n}}^E$ and $p_n^{n+1} := p_{B^{\wedge_E n}}^{B^{\wedge_E n+1}}$.

By Theorem 2.10, $\left(gr_B E, \Delta_{gr_B E}, \varepsilon_{gr_B E} = \varepsilon_C \circ \sigma_C^0 \circ p_0^{gr_C E}\right)$ is a strongly $\mathbb{N}$-graded coalgebra.

By Proposition 4.5, $\left(gr_B E, m_{gr_B E}, u_{gr_B E} = i_0^{gr_B E} \circ p_0^1 \circ u_B\right)$ is a graded algebra with the same graduation defined by $\left(gr_B^k E\right)_{k\in\mathbb{N}}$.

By Lemma 4.1, in order to get that $gr_B E$ is a graded braided bialgebra we have to prove that

$$\sum_{s+t=a+b} \left(i_s^{gr_B E} \otimes i_t^{gr_B E}\right) \circ \Delta_{s,t}^{gr_B E} \circ m_{a,b}^{gr_B E}$$

$$= \sum_{s'+t'=a} \sum_{s''+t''=b} \left[\begin{array}{c} \left(i_{s'+s''}^{gr_B E} \otimes i_{t'+t''}^{gr_B E}\right) \circ \left(m_{s',s''}^{gr_B E} \otimes m_{t',t''}^{gr_B E}\right) \circ \\ \circ \left(\frac{B^{\wedge_E s'+1}}{B^{\wedge_E s'}} \otimes c_{\frac{B^{\wedge_E t'+1}}{B^{\wedge_E t'}}, \frac{B^{\wedge_E s''+1}}{B^{\wedge_E s''}}} \otimes \frac{B^{\wedge_E t''+1}}{B^{\wedge_E t''}}\right) \circ \left(\Delta_{s',t'}^{gr_B E} \otimes \Delta_{s'',t''}^{gr_B E}\right) \end{array} \right],$$

for every $a, b \in \mathbb{N}$. Denote by

$$j_z : \frac{E}{B^{\wedge_E z}} \to \oplus_{i\in\mathbb{N}} \frac{E}{B^{\wedge_E i}}$$

the canonical injection. Since

$$\left(\oplus_{w\in\mathbb{N}} \frac{i_{w+1}}{B^{\wedge_E w}}\right) \circ i_s^{gr_B E} = j_s \circ \frac{i_{s+1}}{B^{\wedge_E s}} \text{ for every } s \in \mathbb{N}$$

and $\oplus_{w\in\mathbb{N}} \frac{i_{w+1}}{B^{\wedge_E w}}$ is a monomorphism (our category is an $AB4$ category [Po, page 53]) and $p_a^{a+1} \otimes p_b^{b+1}$ is an epimorphism, the equality we have to prove is equivalent to

$$\sum_{s+t=a+b} (j_s \otimes j_t) \circ \left(\frac{i_{s+1}}{B^{\wedge_E s}} \otimes \frac{i_{t+1}}{B^{\wedge_E t}}\right) \circ \Delta_{s,t}^{gr_B E} \circ m_{a,b}^{gr_B E} \circ \left(p_a^{a+1} \otimes p_b^{b+1}\right)$$

$$= \sum_{s'+t'=a} \sum_{s''+t''=b} \left[\begin{array}{c} (j_{s'+s''} \otimes j_{t'+t''}) \circ \left(\frac{i_{s'+s''+1}}{B^{\wedge_E s'+s''}} \otimes \frac{i_{t'+t''+1}}{B^{\wedge_E t'+t''}}\right) \circ \left(m_{s',s''}^{gr_B E} \otimes m_{t',t''}^{gr_B E}\right) \circ \\ \circ \left(\frac{B^{\wedge_E s'+1}}{B^{\wedge_E s'}} \otimes c_{\frac{B^{\wedge_E t'+1}}{B^{\wedge_E t'}}, \frac{B^{\wedge_E s''+1}}{B^{\wedge_E s''}}} \otimes \frac{B^{\wedge_E t''+1}}{B^{\wedge_E t''}}\right) \\ \circ \left(\Delta_{s',t'}^{gr_B E} \otimes \Delta_{s'',t''}^{gr_B E}\right) \circ \left(p_a^{a+1} \otimes p_b^{b+1}\right) \end{array} \right]$$

By using (33), (16), (32), the compatibility between Δ_E and m_E, and (29), the first term rewrites as

$$\sum_{s+t=a+b} (j_s \otimes j_t) \circ (p_s \otimes p_t) \circ (m_E \otimes m_E) \circ (E \otimes c_{E,E} \otimes E) \circ (\beta_{a+1} \otimes \beta_{b+1}) \circ (\alpha_{a+1} \otimes \alpha_{b+1})$$

On the other hand, in view of (46), the second term rewrites as $\Xi \circ (\alpha_{a+1} \otimes \alpha_{b+1})$ where

$$\Xi = \sum_{\substack{s'+t'=a \\ s''+t''=b}} \left[\begin{array}{c} (j_{s'+s''} \otimes j_{t'+t''}) \circ \left(\frac{i_{s'+s''+1}}{B^{\wedge_E s'+s''}} \otimes \frac{i_{t'+t''+1}}{B^{\wedge_E t'+t''}}\right) \circ \left(m_{s',s''}^{gr_B E} \otimes m_{t',t''}^{gr_B E}\right) \circ \\ \circ \left(\frac{B^{\wedge_E s'+1}}{B^{\wedge_E s'}} \otimes c_{\frac{B^{\wedge_E t'+1}}{B^{\wedge_E t'}}, \frac{B^{\wedge_E s''+1}}{B^{\wedge_E s''}}} \otimes \frac{B^{\wedge_E t''+1}}{B^{\wedge_E t''}}\right) \circ (\theta_{s',t'} \otimes \theta_{s'',t''+1}) \end{array} \right].$$

We will prove that

$$\Xi = \sum_{s+t=a+b} (j_s \otimes j_t) \circ (p_s \otimes p_t) \circ (m_E \otimes m_E) \circ (E \otimes c_{E,E} \otimes E) \circ (\beta_{a+1} \otimes \beta_{b+1})$$

Since $\gamma_{a+1}\otimes\gamma_{b+1}$ is an epimorphism, equivalently we will prove that

$$\begin{aligned}&\Xi\circ(\gamma_{a+1}\otimes\gamma_{b+1})\\ = &\sum_{s+t=a+b}(j_s\otimes j_t)\circ(p_s\otimes p_t)\circ(m_E\otimes m_E)\circ(E\otimes c_{E,E}\otimes E)\circ(\beta_{a+1}\otimes\beta_{b+1})\circ(\gamma_{a+1}\otimes\gamma_{b+1}).\end{aligned}$$

This is achieved by using (47), (33), (32), naturality of braiding and (30).

In view of Lemma 4.1, it remains to prove that

$$\varepsilon_0^{gr_BE}\circ m_{0,0}^{gr_BE}=m_{\mathbf{1}}\circ\left(\varepsilon_0^{gr_BE}\otimes\varepsilon_0^{gr_BE}\right).$$

This follows easily once proved that $\varepsilon_0^{gr_BE}=\varepsilon_B$ and $m_{0,0}^{gr_BE}=m_B$.

In view of Theorem 2.10, we have

$$\varepsilon_0^{gr_BE}=\varepsilon_{gr_CE}\circ i_0^{gr_BE}=\varepsilon_E\circ\sigma_B^0\circ p_0^{gr_BE}\circ i_0^{gr_BE}=\varepsilon_E\circ i_B^E=\varepsilon_B.$$

Since

$$i_B^E\circ m_\wedge^{0,0}\overset{(32)}{=}m_E\left(i_B^E\otimes i_B^E\right)=i_B^Em_B$$

we get that $m_\wedge^{0,0}=m_B$ and hence

$$m_{0,0}^{gr_BE}=m_{0,0}^{gr_BE}\circ\left(p_0^1\otimes p_0^1\right)\overset{(33)}{=}p_0^1\circ m_\wedge^{0,0}=m_\wedge^{0,0}=m_B.$$

□

THEOREM 4.7. *Let $(\mathcal{M},c)$ be a cocomplete and complete abelian coabelian braided monoidal category satisfying $AB5$. Assume that the tensor product commutes with direct sums. Let $i_B^E:B\hookrightarrow E$ be a monomorphism in $\mathcal{M}$ which is a braided bialgebra homomorphism in $\mathcal{M}$ and let $i_{B^{\wedge_E n}}^E:B^{\wedge_E n}\to E$ and $i_{B^{\wedge_E a}}^{B^{\wedge_E b}}:B^{\wedge_E a}\to B^{\wedge_E b}$ $(a\leq b)$ be the canonical injections. Assume that $i_{B^{\wedge_E a}}^{B^{\wedge_E b}}$ is a split monomorphism in $\mathcal{M}$ for every $a\leq b$. The following assertions are equivalent.*

(1) *gr_BE is the braided bialgebra of type one associated to B and $\frac{B\wedge_EB}{B}$.*
(2) *gr_BE is strongly $\mathbb{N}$-graded as an algebra.*
(3) *$A=\oplus_{n\in\mathbb{N}}B^{\wedge_E n+1}$ is strongly $\mathbb{N}$-graded as an algebra.*
(4) *$B^{\wedge_E n+1}=\left(B^{\wedge_E 2}\right)^{\cdot_E n}$ for every $n\geq 2$.*

Proof. Consider the graded algebra homomorphism

$$\oplus_{n\in\mathbb{N}}p_{B^{\wedge_E n}}^{B^{\wedge_E n+1}}:\oplus_{n\in\mathbb{N}}B^{\wedge_E n+1}\to gr_BE$$

of Proposition 4.5.

(1) ⇔ (2) It follows in view of [AM1, Theorem 6.8] (where $AB5$ is required) and by Theorem 2.10.

(2) ⇔ (3) It follows by Theorem 3.6.

(3) ⇔ (4) Let $\varphi:T=T_B\left(B^{\wedge_E 2}\right)\to\oplus_{n\in\mathbb{N}}B^{\wedge_E n+1}$ be the canonical morphism arising from the universal property of the tensor algebra and let $\varphi_n:\left(B^{\wedge_E 2}\right)^{\otimes_B n}\to B^{\wedge_E n+1}$ be its graded n-th component. In view of [AM1, Theorem 3.11], (3) is equivalent to require that φ_n is an epimorphism for every $n\geq 2$ (note that φ_0 and φ_1 are always isomorphisms). Let us prove that

$$\varphi_n\circ\chi_{\left(B^{\wedge_E 2}\right)^{\otimes_B n-1},B^{\wedge_E 2}}=m_\wedge^{n-1,1}\circ\left(\varphi_{n-1}\otimes B^{\wedge_E 2}\right),\ \text{for every } n\geq 2. \tag{34}$$

where $\chi_{X,Y}:X\otimes Y\to X\otimes_B Y$ denotes the canonical projection.

Note that, being φ a graded homomorphism and in view of its definition, one has

$$i_{B^{\wedge_E n}}^A\circ\varphi_{n-1}=\varphi\circ i_{\left(B^{\wedge_E 2}\right)^{\otimes_B n-1}}^T=\overline{m}_A^{n-2}\circ\left(i_{B^{\wedge_E 2}}^A\right)^{\otimes_B n-1}$$

so that

$$\begin{aligned}&\varphi_n\circ\chi_{\left(B^{\wedge_E 2}\right)^{\otimes_B n-1},B^{\wedge_E 2}}=p_n^A\circ\overline{m}_A^{n-1}\circ\left(i_{B^{\wedge_E 2}}^A\right)^{\otimes_B n}\circ\chi_{\left(B^{\wedge_E 2}\right)^{\otimes_B n-1},B^{\wedge_E 2}}\\ = &\ p_n^A\circ m_A\circ\left[\left(\overline{m}_A^{n-2}\circ\left(i_{B^{\wedge_E 2}}^A\right)^{\otimes_B n-1}\right)\otimes i_{B^{\wedge_E 2}}^A\right]\\ = &\ p_n^A\circ m_A\circ\left[\left(i_{B^{\wedge_E n}}^A\circ\varphi_{n-1}\right)\otimes i_{B^{\wedge_E 2}}^A\right]=p_n^A\circ m_A\circ\left(i_{B^{\wedge_E n}}^A\otimes i_{B^{\wedge_E 2}}^A\right)\circ\left(\varphi_{n-1}\otimes B^{\wedge_E 2}\right)\end{aligned}$$

$$\overset{(19)}{=} p_n^A \circ i_{B^{\wedge_E n+1}}^A \circ m_{n-1,1}^A \circ \left(\varphi_{n-1} \otimes B^{\wedge_E 2}\right) = m_{\wedge}^{n-1,1} \circ \left(\varphi_{n-1} \otimes B^{\wedge_E 2}\right).$$

Hence (34) holds. Next we prove

$$i_{B^{\wedge_E n+1}}^E \circ \varphi_n = \overline{m}_E^{n-1} \circ \left(i_{B^{\wedge_E 2}}^E\right)^{\otimes_B n}, \text{ for every } n \geq 2. \tag{35}$$

This is achieved by induction, composing on the right both sides with the epimorphism $\chi_{\left(B^{\wedge_E 2}\right)^{\otimes_B n-1}, B^{\wedge_E 2}}$ and using (34), (32).

Now, if φ_n is an epimorphism, then

$$\left(B^{\wedge_E n+1}, i_{B^{\wedge_E n+1}}^E\right) = \mathrm{Im}\left[\overline{m}_E^{n-1} \circ \left(i_{B^{\wedge_E 2}}^E\right)^{\otimes_B n}\right] = \left(B^{\wedge_E 2}\right)^{\cdot_E n}.$$

Conversely, if $\left(B^{\wedge_E n+1}, i_{B^{\wedge_E n+1}}^E\right) = \left(B^{\wedge_E 2}\right)^{\cdot_E n}$, then there exists an epimorphism

$$\varphi_n' : \left(B^{\wedge_E 2}\right)^{\otimes_B n} \to B^{\wedge_E n+1}$$

such that

$$i_{B^{\wedge_E n+1}}^E \circ \varphi_n' = \overline{m}_E^{n-1} \circ \left(i_{B^{\wedge_E 2}}^E\right)^{\otimes_B n}.$$

In view of (35) and since $i_{B^{\wedge_E n+1}}^E$ is a monomorphism, we get that $\varphi_n = \varphi_n'$. □

Corollary 4.8. *Let H be a subbialgebra of a bialgebra E over a field K. The following assertions are equivalent.*

1. *$gr_H E$ is the bialgebra of type one associated to H and $\frac{H \wedge_E H}{H}$.*
2. *$gr_H E$ is strongly $\mathbb{N}$-graded as an algebra i.e. $gr_H E$ is generated as an algebra by H and $\frac{H \wedge_E H}{H}$.*
3. *$\oplus_{n \in \mathbb{N}} H^{\wedge_E n+1}$ is strongly $\mathbb{N}$-graded as an algebra.*
4. *$H^{\wedge_E n+1} = (H \wedge_E H)^{\cdot_E n}$ for every $n \geq 2$.*

Proof. We apply Theorem 4.7 to the case $(\mathcal{M}, c) = (\mathfrak{Vec}(K), \tau)$ where τ is the canonical flip. □

Remark 4.9. Let H be a subbialgebra of a bialgebra E over a field K and assume that H contains the coradical of E (e.g. E is connected). Assume that one of the conditions of Corollary 4.8 holds. Since H contains the coradical of E, then the filtration $\left(H^{\wedge_E n+1}\right)_{n \in \mathbb{N}}$ is exhaustive hence (4) implies that E is generated as an algebra by $H \wedge_E H$. The converse of this implication seems not to be true in general. Nevertheless we could not find a counterexample.

When E is connected, in Corollary 4.8, we recover part of [Kh, Theorem 3.5] although for ordinary Hopf algebras.

5. The Associated Graded Algebra of a Bialgebra With Respect to a Quotient bialgebra

Lemma 5.1. *Let $(\mathcal{M}, c)$ be a cocomplete coabelian braided monoidal category such that the tensor product commutes with direct sums. Consider in $(\mathcal{M}, c)$ a datum $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ consisting of a graded object B with graduation defined by $(B_k)_{k \in \mathbb{N}}$ such that, with respect to this graduation,*

- *(B, m_B, u_B) is a graded algebra and*
- *$(B, \Delta_B, \varepsilon_B)$ is a graded coalgebra.*

Assume that for every $a, b \in \mathbb{N}$,

$$\sum_{s+t=a+b} \Delta_{a,b} m_{s,t} \left(p_s \otimes p_t\right)$$

$$= \sum_{s'+t'=a} \sum_{s''+t''=b} \left(m_{s',t'} \otimes m_{s'',t''}\right) \circ \left(B_{s'} \otimes c_{B_{s''}, B_{t'}} \otimes B_{t''}\right) \circ \left(\Delta_{s',s''} \otimes \Delta_{t',t''}\right) \circ \left(p_{s'+s''} \otimes p_{t'+t''}\right)$$

$$\Delta_{0,0} \circ u_0 = (u_0 \otimes u_0) \circ \Delta_{\mathbf{1}}.$$

Then B is a graded braided bialgebra in $(\mathcal{M}, c)$.

Proof. It is analogous to that of Lemma 4.1. □

NOTATIONS 5.2. *From now on the following assumptions and notations will be used.*

$(\mathcal{M}, c)$ *is a cocomplete abelian coabelian braided monoidal category. Assume that the tensor product commutes with direct sums.*

Let $\pi : E \to B$ *be an epimorphism in* $\mathcal{M}$ *which is a braided bialgebra homomorphism in* $\mathcal{M}$ *and let* $\left(I, i_I^E\right) := \ker(\pi)$. *Assume that*

$$\frac{E}{i_{I^{a+1}}^{I^a}} : \frac{E}{I^{a+1}} \to \frac{E}{I^a}$$

is a split epimorphism for every $a \in \mathbb{N}$, *where* $i_{I^{a+1}}^{I^a} : I^{a+1} \to I^a$ *is the canonical injection.*

The family $((\frac{E}{I^a})_{a\in\mathbb{N}}, (\frac{E}{i_{I^{a+1}}^{I^a}})_{a\in\mathbb{N}})$ *fulfills the conditions of Theorem 4.2 when regarded inside the dual of the abelian monoidal category* $\mathcal{M}$. *Thus we have the following exact sequence*

$$\bigoplus_{a+b=n} I^a \otimes I^b \overset{\nabla\left[\left(i_{I^a}^E \otimes i_{I^b}^E\right)_{a+b=n}\right]}{\longrightarrow} E \otimes E \overset{\Delta\left[\left(p_{I^a}^E \otimes p_{I^b}^E\right)_{a+b=n+1}\right]}{\longrightarrow} \bigoplus_{a+b=n+1} \frac{E}{I^a} \otimes \frac{E}{I^b}.$$

Let

$$(I_n(E), \beta_n) := \mathrm{Im}\left\{\Delta\left[\left(p_{I^a}^E \otimes p_{I^b}^E\right)_{a+b=n+1}\right]\right\}$$

and let

$$\gamma_n : E \otimes E \twoheadrightarrow I_n(E)$$

be the unique morphism such that

$$\beta_n \circ \gamma_n = \Delta\left[\left(p_{I^a}^E \otimes p_{I^b}^E\right)_{a+b=n+1}\right]. \tag{36}$$

Since $\left(I^{a+b}, i_{I^{a+b}}^E\right) = \mathrm{Im}\left(m_E \circ \left(i_{I^a}^E \otimes i_{I^b}^E\right)\right)$, *we have* $p_{I^{a+b}}^E \circ m_E \circ \left(i_{I^a}^E \otimes i_{I^b}^E\right) = 0$ *so that*

$$p_{I^n}^E \circ m_E \circ \nabla\left[\left(i_{I^a}^E \otimes i_{I^b}^E\right)_{a+b=n}\right] = 0$$

and hence, by the exactness of the sequence above, there exists a unique morphism

$$\alpha_n : I_n(E) \to \frac{E}{I^n}$$

such that

$$p_{I^n}^E \circ m_E = \alpha_n \circ \gamma_n, \text{ for every } n \in \mathbb{N}. \tag{37}$$

LEMMA 5.3. *0) For every* $s, t, u, v \in \mathbb{N}$, *we have*

$$(\beta_u \otimes \beta_v) \circ (\gamma_u \otimes \gamma_v) \circ (E \otimes c \otimes E) \circ (\Delta_E \otimes \Delta_E) \circ \left(i_{I^s}^E \otimes i_{I^t}^E\right) \tag{38}$$

$$= \Delta\left[\left(\left(\frac{E}{I^a} \otimes c_{\frac{E}{I^b}, \frac{E}{I^c}} \otimes \frac{E}{I^d}\right) \circ \left(\left[\left(p_{I^a}^E \otimes p_{I^b}^E\right) \Delta_E i_{I^s}^E\right] \otimes \left[\left(p_{I^c}^E \otimes p_{I^d}^E\right) \Delta_E i_{I^t}^E\right]\right)\right)_{\substack{a+c=u+1\\ b+d=v+1}}\right]$$

1) The following relations hold.

$$\left(p_{I^u}^E \otimes p_{I^v}^E\right) \circ \Delta_E \circ i_{I^{u+v-1}}^E = 0 \text{ for every } u, v \in \mathbb{N}, u + v \geq 1. \tag{39}$$

2) For every $a, b \in \mathbb{N}$, *there exists a unique morphism*

$$\Delta_\vee^{a,b} : \frac{E}{I^{a+b+1}} \to \frac{E}{I^{a+1}} \otimes \frac{E}{I^{b+1}}$$

such that

$$\Delta_\vee^{a,b} \circ p_{I^{a+b+1}}^E = \left(p_{I^{a+1}}^E \otimes p_{I^{b+1}}^E\right) \Delta_E, \text{ for every } a, b \in \mathbb{N}. \tag{40}$$

3) For every $a, b, c, d \in \mathbb{N}$, *we have*

$$\left(\Delta_\vee^{a,b} \otimes \frac{E}{I^{c+1}}\right) \circ \Delta_\vee^{a+b,c} = \left(\frac{E}{I^{a+1}} \otimes \Delta_\vee^{b,c}\right) \circ \Delta_\vee^{a,b+c},$$

$$\left(\frac{E}{I^{d+1}} \otimes \varepsilon_B\right) \circ \Delta_\vee^{d,0} = r_{E/I^{d+1}}^{-1}, \qquad \left(\varepsilon_B \otimes \frac{E}{I^{d+1}}\right) \circ \Delta_\vee^{0,d} = l_{E/I^{d+1}}^{-1},$$

$$\left(\frac{E}{i_{I^{a+2}}^{I^{a+1}}}\otimes\frac{E}{I^{b+1}}\right)\circ\Delta_{\vee}^{a+1,b}=\Delta_{\vee}^{a,b}\circ\frac{E}{i_{I^{a+b+2}}^{I^{a+b+1}}},\qquad\left(\frac{E}{I^{a+1}}\otimes\frac{E}{i_{I^{b+2}}^{I^{b+1}}}\right)\circ\Delta_{\vee}^{a,b+1}=\Delta_{\vee}^{a,b}\circ\frac{E}{i_{I^{a+b+2}}^{I^{a+b+1}}}$$

Proof. It is analogous to that of Lemma 4.4. □

PROPOSITION 5.4. *$\oplus_{n\in\mathbb{N}}\frac{E}{I^{n+1}}$ is a graded coalgebra and there is a unique coalgebra structure on $gr_I E=\oplus_{n\in\mathbb{N}}gr_I^n E$ such that*

$$\oplus_{n\in\mathbb{N}}\frac{i_{I^n}^E}{I^{n+1}}:gr_I E\to\oplus_{n\in\mathbb{N}}\frac{E}{I^{n+1}}$$

is a coalgebra homomorphism and

(1) *$gr_I E=\oplus_{n\in\mathbb{N}}gr_I^n E$ is a graded coalgebra such that $\oplus_{n\in\mathbb{N}}\frac{i_{I^n}^E}{I^{n+1}}$ is a graded homomorphism;*
(2)

$$\left(\frac{i_{I^a}^E}{I^{a+1}}\otimes\frac{i_{I^b}^E}{I^{b+1}}\right)\circ\Delta_{a,b}^{gr_I E}=\Delta_{\vee}^{a,b}\circ\frac{i_{I^{a+b}}^E}{I^{a+b+1}};\tag{41}$$

(3) *$\varepsilon_{gr_I E}=\varepsilon_B\circ p_0^{gr_I E}$.*

Proof. By Lemma 5.3, we can apply the Lemma 2.5 to the family $(\frac{E}{I^{n+1}})_{n\in\mathbb{N}}$. □

THEOREM 5.5. *Let $(\mathcal{M},c)$ be a cocomplete and complete abelian coabelian braided monoidal category satisfying $AB5$. Assume that the tensor product commutes with direct sums.*

Let $\pi:E\to B$ be an epimorphism in $\mathcal{M}$ which is a braided bialgebra homomorphism in $\mathcal{M}$ and let $\left(I,i_I^E\right):=\ker(\pi)$. Assume that

$$\frac{E}{i_{I^{a+1}}^{I^a}}:\frac{E}{I^{a+1}}\to\frac{E}{I^a}$$

is a split epimorphism for every $a\in\mathbb{N}$, where $i_{I^{a+1}}^{I^a}:I^{a+1}\to I^a$ is the canonical injection. Then $gr_I E$ is a graded braided bialgebra in $(\mathcal{M},c)$.

Proof. It is analogous to that of Theorem 4.6. □

THEOREM 5.6. *Let $(\mathcal{M},c)$ be a cocomplete and complete abelian coabelian braided monoidal category satisfying $AB5$. Assume that the tensor product commutes with direct sums. Let $\pi:E\to B$ be an epimorphism in $\mathcal{M}$ which is a braided bialgebra homomorphism in $\mathcal{M}$ and let $\left(I,i_I^E\right):=\ker(\pi)$. Assume that*

$$\frac{E}{i_{I^{a+1}}^{I^a}}:\frac{E}{I^{a+1}}\to\frac{E}{I^a}$$

is a split epimorphism for every $a\in\mathbb{N}$, where $i_{I^{a+1}}^{I^a}:I^{a+1}\to I^a$ is the canonical injection. The following assertions are equivalent.

(1) *$gr_I E$ is the braided bialgebra of type one associated to $B=\frac{E}{I}$ and $\frac{I}{I^2}$.*
(2) *$gr_I E$ is strongly $\mathbb{N}$-graded as a coalgebra.*
(3) *$C=\oplus_{n\in\mathbb{N}}\frac{E}{I^{n+1}}$ is strongly $\mathbb{N}$-graded as a coalgebra.*
(4) *$\left(I^{n+1},i_{I^{n+1}}^E\right)=\left(I^2\right)^{\wedge_E n}$ for every $n\geq 2$.*

Proof. Consider the graded coalgebra homomorphism

$$\oplus_{n\in\mathbb{N}}\frac{i_{I^n}^E}{I^{n+1}}:gr_I E\to\oplus_{n\in\mathbb{N}}\frac{E}{I^{n+1}}$$

of Proposition 5.4. Note that in view of $AB5$ condition, this morphism is indeed a monomorphism.

(1) ⇔ (2) It follows in view of [AM1, Theorem 6.8] (where $AB5$ is required) and by Theorem 3.10.

(2) ⇔ (3) It follows by Theorem 2.7 that, by Lemma 5.3, can be applied to the family $(\frac{E}{I^{n+1}})_{n\in\mathbb{N}}$.

(3) ⇔ (4) Let $\psi:\oplus_{n\in\mathbb{N}}\frac{E}{I^{n+1}}\to T^c=T^c_{\frac{E}{I}}\left(\frac{E}{I^2}\right)$ be the canonical morphism arising from the universal property of the tensor algebra and let

$$\psi_n:\frac{E}{I^{n+1}}\to\left(\frac{E}{I^2}\right)^{\square_B n}$$

be its graded n-th component. In view of [AM1, Theorem 2.22], (3) is equivalent to require that ψ_n is an epimorphism for every $n \geq 2$ (note that ψ_0 and ψ_1 are always isomorphisms). Let us prove that

$$\zeta_{\left(\frac{E}{I^2}\right)^{\square_B n-1}, \frac{E}{I^2}} \circ \psi_n = \left(\psi_{n-1} \otimes \frac{E}{I^2}\right) \circ \Delta_{\vee}^{n-1,1}, \text{ for every } n \geq 2. \tag{42}$$

where $\zeta_{X,Y} : X \square_B Y \to X \otimes Y$ denotes the canonical injection.

Note that, being ψ a graded homomorphism and in view of [AM1, Theorem 2.16 and Proposition 2.19], one has

$$\psi_n \circ p_{n-1}^C = p_{n-1}^{T^c} \circ \psi = \left(p_1^C\right)^{\square_B n-1} \circ \overline{\Delta}_C^{n-2}$$

so that

$$\begin{aligned}
& \zeta_{\left(\frac{E}{I^2}\right)^{\square_B n-1}, \frac{E}{I^2}} \circ \psi_n \\
= \; & \zeta_{\left(\frac{E}{I^2}\right)^{\square_B n-1}, \frac{E}{I^2}} \circ \left(p_1^C\right)^{\square_B n} \circ \overline{\Delta}_C^{n-1} \circ i_n^C = \left[\left(p_1^C\right)^{\square_B n-1} \circ \overline{\Delta}_C^{n-2} \otimes p_1^C\right] \circ \Delta_C \circ i_n^C \\
= \; & \left[\left(\psi_{n-1} \circ p_{n-1}^C\right) \otimes p_1^C\right] \circ \Delta_C \circ i_n^C = \left(\psi_{n-1} \otimes C_1\right) \circ \left(p_{n-1}^C \otimes p_1^C\right) \circ \Delta_C \circ i_n^C \\
& \overset{(5)}{=} \left(\psi_{n-1} \otimes \frac{E}{I^2}\right) \circ \Delta_{n-1,1}^C \circ p_n^C \circ i_n^C = \left(\psi_{n-1} \otimes \frac{E}{I^2}\right) \circ \Delta_{\vee}^{n-1,1}.
\end{aligned}$$

Hence (42) holds. Let us prove by induction that

$$\psi_n \circ p_{I^{n+1}}^E = \left(p_{I^2}^E\right)^{\square_B n} \circ \overline{\Delta}_E^{n-1}, \text{ for every } n \geq 2. \tag{43}$$

$n = 2$) We have

$$\begin{aligned}
& \zeta_{\frac{E}{I^2}, \frac{E}{I^2}} \circ \psi_2 \circ p_{I^3}^E \overset{(42)}{=} \left(\psi_1 \otimes \frac{E}{I^2}\right) \circ \Delta_{\vee}^{1,1} \circ p_{I^3}^E \\
= \; & \Delta_{\vee}^{1,1} \circ p_{I^3}^E \overset{(40)}{=} \left(p_{I^2}^E \otimes p_{I^2}^E\right) \circ \Delta_E = \zeta_{\frac{E}{I^2}, \frac{E}{I^2}} \circ \left(p_{I^2}^E \square_B p_{I^2}^E\right) \circ \overline{\Delta}_E
\end{aligned}$$

$n-1 \Rightarrow n$) We have

$$\begin{aligned}
& \zeta_{\left(\frac{E}{I^2}\right)^{\square_B n-1}, \frac{E}{I^2}} \circ \psi_n \circ p_{I^{n+1}}^E \overset{(42)}{=} \left(\psi_{n-1} \otimes \frac{E}{I^2}\right) \circ \Delta_{\vee}^{n-1,1} \circ p_{I^{n+1}}^E \\
& \overset{(40)}{=} \left(\psi_{n-1} \otimes \frac{E}{I^2}\right) \circ \left(p_{I^n}^E \otimes p_{I^2}^E\right) \circ \Delta_E \\
= \; & \left(\left(p_{I^2}^E\right)^{\square_B n-1} \otimes p_{I^2}^E\right) \circ \left(\overline{\Delta}_E^{n-2} \otimes \frac{E}{I^2}\right) \circ \Delta_E = \zeta_{\left(\frac{E}{I^2}\right)^{\square_B n-1}, \frac{E}{I^2}} \circ \left(p_{I^2}^E\right)^{\square_B n} \circ \overline{\Delta}_E^{n-1}.
\end{aligned}$$

We have so proved that (43) holds.

Now, if ψ_n is a monomorphism, then

$$\left(I^{n+1}, i_{I^{n+1}}^E\right) = \ker\left(\psi_n \circ p_{I^{n+1}}^E\right) = \ker\left[\left(p_{I^2}^E\right)^{\square_B n} \circ \overline{\Delta}_E^{n-1}\right] = \ker\left[\left(p_{I^2}^E\right)^{\otimes n} \circ \Delta_E^{n-1}\right] = \left(I^2\right)^{\wedge_E n}.$$

Conversely, if $\left(I^{n+1}, i_{I^{n+1}}^E\right) = \left(I^2\right)^{\wedge_E n}$, then $\left(p_{I^2}^E\right)^{\square_B n} \circ \overline{\Delta}_E^{n-1}$ factors to a monomorphism

$$\psi_n' : \frac{E}{I^{n+1}} \to \left(\frac{E}{I^2}\right)^{\square_B n}$$

such that $\psi_n' \circ p_{I^{n+1}}^E = \left(p_{I^2}^E\right)^{\square_B n} \circ \overline{\Delta}_E^{n-1}$. In view of (43) and since $p_{I^{n+1}}^E$ is an epimorphism, we get that $\psi_n = \psi_n'$. □

Corollary 5.7. *Let J an ideal of a bialgebra E over a field K and assume that J is also a coideal. The following assertions are equivalent.*

1. *$gr_J E$ is the bialgebra of type one associated to $\frac{E}{J}$ and $\frac{J}{J^2}$.*
2. *$gr_J E$ is strongly $\mathbb{N}$-graded as a coalgebra.*
3. *$\oplus_{n \in \mathbb{N}} \frac{E}{J^{n+1}}$ is strongly $\mathbb{N}$-graded as a coalgebra.*
4. *$J^{n+1} = \left(J^2\right)^{\wedge_E n}$ for every $n \geq 2$.*

Proof. We apply Theorem 5.6 to the case $(\mathcal{M},c)=(\mathfrak{Vec}\left(K\right),\tau)$ where τ is the canonical flip. $\square$

Appendix A. Technicalities

Proof of Lemma 4.4. . Set $i_n:=i^E_{B^{\wedge_E a}}$ and $p_n:=p^E_{B^{\wedge_E n}}$.

0) It follows by using (28) and naturality of c.

1) Let us prove by induction on $u+v\geq 1$ that $p_{u+v-1}m_E\left(i_u\otimes i_v\right)=0$.

$u+v=1$) is trivial as $i_0=0$.

$u+v>1$) If $u=0$ or $v=0$ there is nothing to prove. Let $u,v>0$. Assume that the statement is true for every i,j such that $1\leq i+j<u+v$ and let us prove it for $u+v$. First of all we will prove that

$$(p_{u-1}\otimes p_v)\circ\Delta_E\circ m_E\circ(i_u\otimes i_v)=0. \tag{44}$$

Using the compatibility of Δ_E and m_E, and (29) we get

$$\begin{aligned}&(p_{u-1}\otimes p_v)\circ\Delta_E\circ m_E\circ(i_u\otimes i_v)\\ =\ &(p_{u-1}\otimes p_v)\circ(m_E\otimes m_E)\circ(E\otimes c\otimes E)\circ(\beta_u\otimes\beta_v)\circ(\alpha_u\otimes\alpha_v)\end{aligned}$$

Let us prove that

$$(p_{u-1}\otimes p_v)\circ(m_E\otimes m_E)\circ(E\otimes c\otimes E)\circ(\beta_u\otimes\beta_v)=0.$$

Since $\gamma_u\otimes\gamma_v$ is an epimorphism, this is equivalent to prove that

$$(p_{u-1}\otimes p_v)\circ(m_E\otimes m_E)\circ(E\otimes c\otimes E)\circ(\beta_u\otimes\beta_v)\circ(\gamma_u\otimes\gamma_v)=0.$$

We have

$$\begin{aligned}&(p_{u-1}\otimes p_v)\circ(m_E\otimes m_E)\circ(E\otimes c\otimes E)\circ(\beta_u\otimes\beta_v)\circ(\gamma_u\otimes\gamma_v)\\ &\overset{(30)}{=}\nabla\left[\left(([p_{u-1}m_E\left(i_a\otimes i_b\right)]\otimes[p_vm_E\left(i_c\otimes i_d\right)])\circ\left(B^{\wedge_E a}\otimes c_{B^{\wedge_E c},B^{\wedge_E b}}\otimes B^{\wedge_E d}\right)\right)_{\substack{a+c=u+1\\ b+d=v+1}}\right]\end{aligned}$$

Note that $(a+b)+(c+d)=(a+c)+(b+d)=u+v+2$.

If $u-1\geq a+b-1$ then $a+b\leq u<u+v$ and we have

$$p_{u-1}m_E\left(i_a\otimes i_b\right)=\frac{E}{\xi^{u-1}_{a+b-1}}p_{a+b-1}m_E\left(i_a\otimes i_b\right)=0.$$

If $u-1<a+b-1$, and $v<c+d-1$, then $u+v\leq a+b+c+d-3=u+v-1$. A contradiction.

Then $v\geq c+d-1$. Thus $c+d<v\leq u+v$ so that, as above, we get $p_vm_E\left(i_c\otimes i_d\right)=0$.

Hence

$$(p_{u-1}\otimes p_v)\circ(m_E\otimes m_E)\circ(E\otimes c\otimes E)\circ(\beta_u\otimes\beta_v)\circ(\gamma_u\otimes\gamma_v)=0$$

and so (44) holds.

Let $\Delta_{u-1,v}:=\Delta_{B^{\wedge_E u-1},B^{\wedge_E v}}=(p_{u-1}\otimes p_v)\circ\Delta_E$. Then, as seen in 2.8, there exists a unique morphism

$$\overline{\Delta}_{u-1,v}:\frac{E}{B^{\wedge_E u+v-1}}=\frac{E}{B^{\wedge_E u-1}\wedge_E B^{\wedge_E v}}\to\frac{E}{B^{\wedge_E u-1}}\otimes\frac{E}{B^{\wedge_E v}}$$

such that $\overline{\Delta}_{u-1,v}\circ p_{u+v-1}=\Delta_{u-1,v}$. Furthermore $\overline{\Delta}_{u-1,v}$ is a monomorphism. From

$$\overline{\Delta}_{u-1,v}\circ[p_{u+v-1}\circ m_E\circ(i_u\otimes i_v)]=\Delta_{u-1,v}\circ m_E\circ(i_u\otimes i_v)\overset{(44)}{=}0$$

we deduce $p_{u+v-1}\circ m_E\circ(i_u\otimes i_v)=0$ so that we have proved (31).

2) From (31), we get $p_{a+b+1}m_E\left(i_{a+1}\otimes i_{b+1}\right)=0$ for every $a,b\in\mathbb{N}$. By the universal property of the kernel there exists a unique morphism $m^{a,b}_{\wedge}:B^{\wedge_E a+1}\otimes B^{\wedge_E b+1}\to B^{\wedge_E a+b+1}$ such that (32) holds.

3) From (32), we get

$$\begin{aligned}i_{a+b+c+1}\circ m^{a+b,c}_{\wedge}\circ\left(m^{a,b}_{\wedge}\otimes B^{\wedge_E c+1}\right)&=m_E\left(m_E\otimes E\right)\left(i_{a+1}\otimes i_{b+1}\otimes i_{c+1}\right)\\ i_{a+b+c+1}\circ m^{a,b+c}_{\wedge}\circ\left(B^{\wedge_E a+1}\otimes m^{b,c}_{\wedge}\right)&=m_E\left(E\otimes m_E\right)\left(i_{a+1}\otimes i_{b+1}\otimes i_{c+1}\right)\end{aligned}$$

so that, by associativity of m_E, we obtain

$$i_{a+b+c+1}\circ m_{\wedge}^{a+b,c}\circ\left(m_{\wedge}^{a,b}\otimes B^{\wedge_E c+1}\right)=i_{a+b+c+1}\circ m_{\wedge}^{a,b+c}\circ\left(B^{\wedge_E a+1}\otimes m_{\wedge}^{b,c}\right).$$

Since $i_{a+b+c+1}$ is a monomorphism, we deduce that

$$m_{\wedge}^{a+b,c}\circ\left(m_{\wedge}^{a,b}\otimes B^{\wedge_E c+1}\right)=m_{\wedge}^{a,b+c}\circ\left(B^{\wedge_E a+1}\otimes m_{\wedge}^{b,c}\right).$$

On the other hand, by applying (32), we infer that

$$i_{d+1}\circ m_{\wedge}^{d,0}\circ\left(B^{\wedge_E d+1}\otimes u_B\right)=m_E\left(E\otimes u_E\right)\left(i_{d+1}\otimes\mathbf{1}\right)=r_E\left(i_{d+1}\otimes\mathbf{1}\right)=i_{d+1}\circ r_{B^{\wedge_E d+1}}$$

so that, since i_{d+1} is a monomorphism, we obtain $m_{\wedge}^{d,0}\circ\left(B^{\wedge_E d+1}\otimes u_B\right)=r_{B^{\wedge_E d+1}}$. Similarly we prove that $m_{\wedge}^{0,d}\circ\left(u_B\otimes B^{\wedge_E d+1}\right)=l_{B^{\wedge_E d+1}}$. We have

$$\begin{aligned}&i_{a+b+2}\circ m_{\wedge}^{a+1,b}\circ\left(i_{B^{\wedge_E a+1}}^{B^{\wedge_E a+2}}\otimes B^{\wedge_E b+1}\right)=m_E\circ\left(i_{a+2}\otimes i_{b+1}\right)\circ\left(i_{B^{\wedge_E a+1}}^{B^{\wedge_E a+2}}\otimes B^{\wedge_E b+1}\right)\\ =\;& m_E\circ\left(i_{a+1}\otimes i_{b+1}\right)=i_{a+b+1}\circ m_{\wedge}^{a,b}=i_{a+b+2}\circ i_{B^{\wedge_E a+b+1}}^{B^{\wedge_E a+b+2}}\circ m_{\wedge}^{a,b}.\end{aligned}$$

Since i_{a+b+2} is a monomorphism, we deduce that $m_{\wedge}^{a+1,b}\circ\left(i_{B^{\wedge_E a+1}}^{B^{\wedge_E a+2}}\otimes B^{\wedge_E b+1}\right)=i_{B^{\wedge_E a+b+1}}^{B^{\wedge_E a+b+2}}\circ m_{\wedge}^{a,b}$. The right hand version of this formula follows by similar arguments. □

LEMMA A.1. *There exists a unique morphism*

$$\theta_{a,b}:\sum_{u+v=a+b+2}B^{\wedge_E u}\otimes B^{\wedge_E v}\to\frac{B^{\wedge_E a+1}}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}$$

such that

$$\left(p_{B^{\wedge_E a}}^{B^{\wedge_E a+1}}\otimes p_{B^{\wedge_E b}}^{B^{\wedge_E b+1}}\right)\circ\beta_{a+b+1}=\left(\frac{i_{B^{\wedge_E a+1}}^{E}}{B^{\wedge_E a}}\otimes\frac{i_{B^{\wedge_E b+1}}^{E}}{B^{\wedge_E b}}\right)\circ\theta_{a,b}.\tag{45}$$

Moreover, for every $a,b\in\mathbb{N}$, we have

$$\theta_{a,b}\circ\alpha_{a+b+1}=\Delta_{a,b}^{gr_B E}\circ p_{B^{\wedge_E a+b}}^{B^{\wedge_E a+b+1}},\tag{46}$$

$$\theta_{a,b}\circ\gamma_{a+b+1}=\nabla\left[\left(\delta_{u,a+1}\delta_{v,b+1}p_{B^{\wedge_E a}}^{B^{\wedge_E a+1}}\otimes p_{B^{\wedge_E b}}^{B^{\wedge_E b+1}}\right)_{u+v=a+b+2}\right].\tag{47}$$

Proof. Set $i_n:=i_{B^{\wedge_E a}}^{E}$, $p_n:=p_{B^{\wedge_E n}}^{E}$ and $p_n^{n+1}:=p_{B^{\wedge_E n}}^{B^{\wedge_E n+1}}$.

Since $p_s i_t=0$ for every $s\geq t$, we have

$$\left(\frac{E}{B^{\wedge_E a}}\otimes\frac{E}{\xi_b^{b+1}}\right)\circ\left(p_a\otimes p_b\right)\circ\beta_{a+b+1}\circ\gamma_{a+b+1}\overset{(28)}{=}\nabla\left[\left[\left(p_a i_u\otimes p_{b+1}i_v\right)\right]_{u+v=a+b+2}\right]=0.$$

Since γ_{a+b+1} is an epimorphism, we get

$$\left(\frac{E}{B^{\wedge_E a}}\otimes\frac{E}{\xi_b^{b+1}}\right)\circ\left(p_a\otimes p_b\right)\circ\beta_{a+b+1}=0.\tag{48}$$

By the universal property of the kernel applied to the exact sequence,

$$0\to\frac{E}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}\overset{\frac{E}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}}{\longrightarrow}\frac{E}{B^{\wedge_E a}}\otimes\frac{E}{B^{\wedge_E b}}\overset{\frac{E}{B^{\wedge_E a}}\otimes\frac{E}{\xi_b^{b+1}}}{\longrightarrow}\frac{E}{B^{\wedge_E a}}\otimes\frac{E}{B^{\wedge_E b+1}}\to 0.$$

there exists a unique morphism

$$\theta'_{a,b}:\sum_{u+v=a+b+2}B^{\wedge_E u}\otimes B^{\wedge_E v}\to\frac{E}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}$$

such that

$$\left(\frac{E}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\theta'_{a,b}=\left(p_a\otimes p_b\right)\circ\beta_{a+b+1}.$$

Now

$$\left(\frac{E}{B^{\wedge_E a+1}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\left(\frac{E}{\xi_a^{a+1}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}\right)\circ\theta'_{a,b}=\left(\frac{E}{\xi_a^{a+1}}\otimes\frac{E}{B^{\wedge_E b}}\right)\circ(p_a\otimes p_b)\circ\beta_{a+b+1}=0$$

where the last equality follows analogously to (48). Since $\frac{E}{B^{\wedge_E a+1}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}$ is a monomorphism we get $\left(\frac{E}{\xi_a^{a+1}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}\right)\circ\theta'_{a,b}=0$. By the universal property of the kernel applied to the exact sequence,

$$0\to\frac{B^{\wedge_E a+1}}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}\overset{\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}}{\longrightarrow}\frac{E}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}\overset{\frac{E}{\xi_a^{a+1}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}}{\longrightarrow}\frac{E}{B^{\wedge_E a+1}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}.$$

there exists a unique morphism

$$\theta_{a,b}:\sum_{u+v=a+b+2}B^{\wedge_E u}\otimes B^{\wedge_E v}\to\frac{B^{\wedge_E a+1}}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}$$

such that

$$\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{B^{\wedge_E b+1}}{B^{\wedge_E b}}\right)\circ\theta_{a,b}=\theta'_{a,b}.$$

Thus (45) holds true.

Let us prove (46). We have

$$\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\theta_{a,b}\circ\alpha_{a+b+1}=(p_a\otimes p_b)\circ\beta_{a+b+1}\circ\alpha_{a+b+1}=(p_a\otimes p_b)\circ\Delta_E\circ i_{a+b+1}.$$

On the other hand, in view of definition of $\Delta^B_{a,b}$ (see 2.8), we have

$$\begin{aligned}&\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\Delta^{gr_B E}_{a,b}\circ p^{a+b+1}_{a+b}\overset{(16)}{=}\Delta^B_{a,b}\circ\frac{i_{a+b+1}}{B^{\wedge_E a+b}}\circ p^{a+b+1}_{a+b}\\ =\;&\Delta^B_{a,b}\circ p_{a+b}\circ i_{a+b+1}=(p_a\otimes p_b)\circ\Delta_E\circ i_{a+b+1}.\end{aligned}$$

Hence

$$\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\theta_{a,b}\circ\alpha_{a+b+1}=\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\Delta^{gr_B E}_{a,b}\circ p^{a+b+1}_{a+b}.$$

Since $\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}$ is a monomorphism, we get (46).

Let us prove (47). We have

$$\begin{aligned}&\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\theta_{a,b}\circ\gamma_{a+b+1}\overset{(45)}{=}(p_a\otimes p_b)\circ\beta_{a+b+1}\circ\gamma_{a+b+1}\\ &\overset{(28)}{=}(p_a\otimes p_b)\circ\nabla\left[(i_u\otimes i_v)_{u+v=a+b+2}\right]=\nabla\left[(p_ai_u\otimes p_bi_v)_{u+v=a+b+2}\right]\\ =\;&\nabla\left[\delta_{u,a+1}\delta_{v,b+1}\left(p_ai_{a+1}\otimes p_bi_{b+1}\right)_{u+v=a+b+2}\right]\\ =\;&\left(\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}\right)\circ\nabla\left[\delta_{u,a+1}\delta_{v,b+1}\left(p_a^{a+1}\otimes p_b^{b+1}\right)_{u+v=a+b+2}\right].\end{aligned}$$

Since $\frac{i_{a+1}}{B^{\wedge_E a}}\otimes\frac{i_{b+1}}{B^{\wedge_E b}}$ is a monomorphism we get (47). □

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